The University Tutorial Series.

GENERAL EDITOR: WILLIAM BRIGGS, M.A., LL.B., F.C.S., F.R.A.S.

A TEXT-BOOK OF STATICS.

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BY

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PREFACE.

THE ground covered by this book includes those portions of Statics which are generally read in an elementary course, and for which little or no knowledge of Trigonometry is required. It may, with advantage, be read after the same authors' *Dynamics*, as the latter book gradually leads up to the Parallelogram of Forces, which forms the basis of Statics. But this course is quite optional, for the present book assumes no previous knowledge of Dynamics, and the few dynamical facts and definitions required for the proof of the Parallelogram of Forces are briefly recapitulated in the first chapter.

Every endeavour has been made to remove, as far as possible, all difficulties usually experienced by beginners when reading Statics for the first time. For this reason, hints and explanations have been freely given wherever it seemed desirable, and the important propositions have been profusely illustrated by worked-out examples. In several cases these examples precede instead of following the general investigations, and it is hoped that the reader will thus the better learn how to work out similar examples from first principles instead of merely obtaining their answers by quoting certain formulæ from memory and substituting numerical values in them.

In most text-books on Statics, the Mechanical Powers are collected together in a chapter near the end, and a separate chapter also is usually devoted to "Virtual" Work. After much careful deliberation, we have decided to depart from this plan. The various machines are introduced as soon as the principles on which they depend have been explained, and it is hoped that the earlier chapters have thus been made more interesting. The Principle of Work is freely employed throughout the book as furnishing verifications or alternative proofs of results which have been established independently. Those who wish to omit all considerations relating to Work will have no difficulty in doing so, as the sections in question

have been kept distinct and can readily be picked out by their headings.

Among other special features of the book, we may, perhaps, call attention to the modification of Newton's proof of the Parallelogram of Forces (p. 5), the simple expression for the resultant of two forces including any angle (p. 26), the symmetric conditions of equilibrium of three parallel forces (p. 85), &c.

The general arrangement is the same as in the authors' *Dynamics*. Two sizes of type are used, all the important bookwork being printed in the larger type, while hints, explanations, examples, alternative proofs, and a few of the less important theorems are printed in smaller type. Those articles which deal with the fundamental principles of the subject—such as the Parallelogram of Forces—have their *numbers* as well as their headings printed in dark type (thus —**136**). These the student should be able to reproduce from memory.

It is hoped that the "Summary of Results" at the end of each chapter will prove of use in facilitating a final revision of the whole subject.

Where results are given as *corollaries*, it is not in most cases sufficient in proving them to quote the *result* of the proposition on which they depend: the student will do well to write out complete proofs, employing, as far as desirable, the same *methods* of proof as are used for the propositions themselves.

In the letterpress, letters which denote *points* in figures are printed thus—**P**, **Q**, so that **PQ** will denote a line and **PQR** an angle or a triangle. Letters denoting algebraic *magnitudes*, such as forces, are printed in ordinary italics thus—P, Q, so that PQ and PQR denote products of two and three algebraic quantities, respectively. In a few of the figures this rule has not been adhered to where no confusion is likely to arise.

Our thanks are due to Mr. F. Rosenberg for the great care and trouble he has taken in revising the proofs.

W. B.
G. H. B.

BURLINGTON HOUSE,
CAMBRIDGE;
July, 1894.

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STATICS.

PART I.

EQUILIBRIUM OF FORCES AT A POINT.

CHAPTER I.

THE PARALLELOGRAM OF FORCES.

1. **Statics** is that branch of Mechanics which deals with forces applied to a body or a number of bodies at **rest.**

For the present we shall only consider the properties of forces applied to a single **particle.** In Chap. IV. we shall treat of forces acting on a body of extended size.

2. Force is defined as that which changes or tends to change a body's state of rest or motion.

A **particle** is a body whose size is so small that it may be regarded as a quantity of matter collected at a single point.

3. Two forces are said to be **equal** if, when they are applied to the *same* body for *equal* intervals of time, they tend to impart the same velocity, or change of velocity, to the body.

If one force imparts double the change of velocity that would be imparted on the same body in an equal interval of time by another force, the first force is said to be double the second, and so on. Thus forces are proportional to the velocities they would impart to the same body in equal intervals of time. In this way different forces may be compared, and the **magnitude** of a force may be

measured in terms of any force which is chosen as the standard or "unit of force."

The **direction** of a force is defined as the direction of the velocity which it tends to impart to the body on which it acts.

Thus force is characterized by having both **magnitude** and **direction.**

4. When a force is applied to start a body from **rest,** the *actual distance* traversed in any time is proportional to the force*, and the *actual direction of motion* is the direction of the force. Thus, if a body initially *at rest* is pulled with a force of 2 lbs., it will move in the direction in which it is pulled, and the distance which it will move over in one minute will be twice what it would move over if pulled by a force of 1 lb. for the same time.

5. Equilibrium.—Resultant force.—If two equal forces act on the same particle in exactly opposite directions, the motion which one force tends to impart is exactly the reverse of that which the other tends to impart. The particle cannot move in opposite directions at the same time, and there is no reason why it should move in one direction rather than the other. Hence it will remain at rest, and the two forces will be said to *balance*, or be *in equilibrium.*

When several independent forces are applied simultaneously to a single particle at rest, the particle may either remain at rest or begin to move in some direction.

If it remains at rest, the forces are said to **balance,** or be **in equilibrium.**

If it starts into motion in any direction, this motion must be the same as would be produced by a certain single force of suitable magnitude applied in that direction. This force is called the **resultant** of the several forces.

Conversely, any forces which have a given force for their resultant are called **components** of the given force.

It should not be forgotten, however, that forces are in equilibrium when the body on which they act moves uniformly in a straight line just as well as when it is at rest (*Dynamics*, § 193).

* See *Dynamics*, Chap. VI. This follows at once from the equations
$$P = mf, \quad s = \tfrac{1}{2}ft^2.$$

6. **Systems of forces.**—The forces with which we shall deal in *Statics* will always be supposed to be kept in equilibrium. But it is often necessary to consider the properties of some of the forces apart from the rest. Any number of forces may be called a **system** of forces, and may subsequently be referred to as "the system" to avoid repeatedly specifying what forces are meant.

When a number of forces have a resultant we may reduce them to a system in equilibrium by applying an additional force equal and opposite to that resultant. For the whole system is then equivalent to two equal and opposite forces (viz., the original resultant and the added force), and is therefore in equilibrium.

Conversely, when any number of forces are in equilibrium, each force is equal and opposite to the resultant of all the rest taken together.

7. **Point of application of a force.**—The conditions of equilibrium of a system of forces do not depend on the nature and *mass* of the body on which they act, but only on the forces themselves. We may, therefore, speak of a force as **acting at a point,** meaning a force "applied to any particle placed at that point." And the **point of application** of the force is "the point at which the particle acted on by the force is situated."

These must, however, be regarded merely as abbreviations, for force can only act where there is matter for it to act on.

8. **Statical units of force.**—The forces which occur most frequently in Statics are those due to *weight*. Hence the most convenient statical unit of force is the *weight of a pound*, and this we call "a force of 1 lb." Larger forces may, however, be measured in hundredweights or tons.

If the French system of weights and measures is used, the statical unit of force will be the weight of a gramme or of a kilogramme (1,000 grammes), according to which is the most convenient.

Thus, in Statics, forces are always measured in gravitation units unless otherwise stated.

9. Forces may be represented by straight lines.—In order to completely define a force, it is necessary to specify

(i.) its point of application,
(ii.) its direction,
(iii.) its magnitude.

All these data will be specified by a straight line of finite length, provided that

(i.) the line is drawn from the *point of application* of the force,
(ii.) it is drawn pointing in the *direction* of the force,
(iii.) its *length* is proportional to the *magnitude* of the force.

Such a straight line is said to *represent* the force.

The *sense* of the direction may be shown by an arrow drawn on or by the side of the line, or by the *order* of the letters used in naming the line. Thus AB represents a force acting from A towards B, BA a force acting from B towards A.

10. **On the choice of a scale of representation.**—In order that the length of a straight line may represent the magnitude of a force, the line should properly contain as many units of length as the force contains units of force. Thus, if a line 1 inch long represents a force of 1 lb., a line 2 inches long will represent a force of 2 lbs., and so on. Very often, however, it is necessary to adopt some other scale of representation suggested by the conditions of the problem. *We may so choose the scale of representation that one of the forces is represented by a straight line of any length we please.* When this has been done, a line of double the length will represent double the force, and so on, so that the lines representing all the *other* forces will then be fully determined.

Thus it might be convenient for some reason to agree to represent a force of 7 lbs. by a length of $\frac{1}{4}$ inch. On this scale a length of $\frac{1}{2}$ inch would represent a force of 14 lbs., and so on.

11. It is often necessary to represent a force *in magnitude and direction only* by a straight line *not* drawn from its point of application. A force will be represented to this extent by *any straight line drawn equal and parallel* to the line which fully represents it, for *parallel straight lines are to be regarded as having the same direction.*

12. THE PARALLELOGRAM OF FORCES.—If two forces acting on the same particle be represented by two adjacent sides of a parallelogram, drawn from their point of application, their resultant shall be represented by the diagonal of the parallelogram drawn from that point.

Let two constant forces P, Q be applied to a particle at rest at A, in directions AB, AC, respectively.

Let AB be the distance the particle would traverse in a given time t if it were set in motion by the constant force P alone.

Let AD be the distance traversed in the same time if acted on similarly by Q alone.

Complete the parallelogram $ABCD$.

Then shall AC be the distance traversed by the particle in the time t if set in motion by both forces P, Q acting

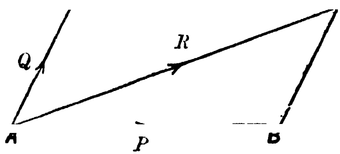


Fig. 1.

on it simultaneously in directions parallel to AB, AD, respectively.

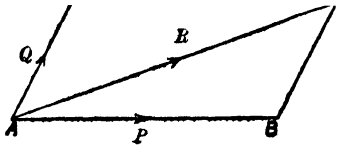
Also AB, AD shall represent the forces P, Q, and AC shall represent their resultant.

(i.) For, since the force Q always acts parallel to the line BC, it can have no effect in changing the rate at which the particle approaches BC in consequence of the other force P. Therefore the particle will reach the line BC in the same time, whether the force Q be applied or not. Therefore at the end of the time t it must be *somewhere* in the line BC.

Similarly it must be somewhere in the line DC.

Therefore at the end of the time t the particle must be at C, the intersection of AC and DC.

Therefore AC is the distance actually traversed in the time t.



(ii.) Hence the resultant of P and Q must be that force which would move the particle from rest along AC in the time t. It must therefore act in the direction AC.

And since the distance traversed in the given time t by a particle starting from rest is proportional to the force acting on it (§ 4), therefore AB, AD, AC are proportional to P, Q and their resultant.

Therefore AD represents Q on the same scale that AB represents P, also AC represents the resultant of P and Q on the same scale.

*Therefore the resultant is represented by the diagonal of the parallelogram whose sides represent the two forces.** [Q.E.D.]

13. Experimental verification of the Parallelogram of Forces.

(a) MECHANICAL DETAILS. — Take three strings; knot them together in a point. To their ends attach any three weights P, Q, R, say P, Q, R lbs., respectively (any two of which are together greater than the third). Allow one string to hang freely with its suspended weight R, and pass the other two over two smooth pulleys H, K, fixed in front of a vertical wall (Fig. 2).

(b) GEOMETRICAL CONSTRUCTION. — When the strings have taken up a position of equilibrium with the knot at

* The above proof is an adaptation of Newton's original proof to the case of constant forces. The usual "dynamical" proof is given in *Dynamics*, § 171.

A, measure off along AH, AK lengths AB, AD, containing P and Q units of length, respectively. On the wall complete the parallelogram $ABCD$, and join AC.

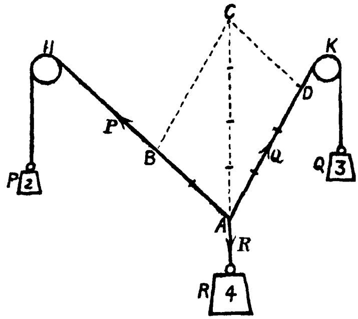


Fig. 2.

(*c*) OBSERVED FACTS.—*Then it will be invariably found*

(i.) *that the diagonal AC is vertical,*

(ii.) *that AC contains R units of length.*

(*d*) DEDUCTIONS.—Now the knot A is in equilibrium under the pulls P, Q, R acting along the strings, respectively. Therefore the resultant of P, Q is equal and opposite to the weight R.

Therefore it is a force of R lbs. acting vertically upwards.

But AC is vertical,

$\therefore$ AC represents the resultant in direction

Also AC contains R units of length;

$\therefore$ AC represents the resultant in magnitude.

But AB, AD represent the forces P, Q.

Therefore the diagonal of the parallelogram represents the resultant of the two forces which are represented separately by its sides. [Q.E.D.]

14. **Fig.** 2 is drawn for the case in which $P = 2$ lbs., $Q = 3$ lbs., $R = 4$ lbs. The measured lengths AB, AD must therefore contain 2 and 3 units of length respectively. When the parallelogram is constructed, the diagonal AC will be found to be vertical and to contain 4 units of length.

OBSERVATION. — Each experiment proves the Parallelogram of Forces to hold good for one particular set of forces only. To give a satisfactory proof it would be necessary to perform a large number of such experiments, using different arrangements of weights each time.

Example.—To find the resultant of forces of 7 lbs. and 11 lbs., whose directions include an angle of 60°.

Take any unit of length and measure off AB, AD containing 7 and 11 units respectively, making $\angle BAD = 60°$. Complete the parallelogram $ABCD$.

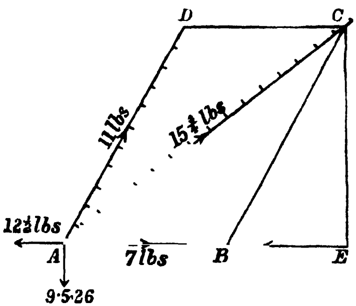


Fig. 3.

Then AC represents the resultant.

On AC mark off from A a scale of the selected units. Then C will be found to lie between the 15th and 16th marks, so that AC contains about $15\frac{3}{4}$ units.

Therefore the resultant force $= 15\frac{3}{4}$ lbs. wt. roughly.

15. *Half the parallelogram is sufficient.* Since BC is equal and parallel to AD, it represents the force Q in magnitude and direction, but not in position (as in § 11). Hence, if two forces acting on a particle are represented in magnitude and direction *only* by two sides of a triangle, AB, BC, their resultant is represented in magnitude and direction by the third side AC.

16. The Triangle of Forces.—*If three forces acting on the same particle can be represented in magnitude and direction (but not in position) by the sides of a triangle taken in order,* they shall be in equilibrium.*

Let three forces P, Q, R, acting on the same particle at O, be represented in magnitude and direction (but not

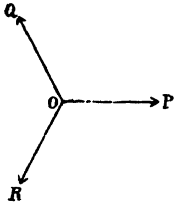


Fig. 4.

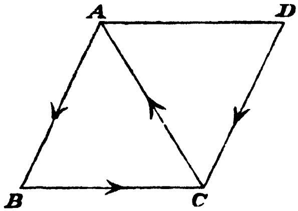


Fig. 5.

in position) by the sides BC, CA, AB of the triangle ABC, respectively.

Then shall the forces be in equilibrium.

Complete the parallelogram $ABCD$.

Then the forces R, P are represented in magnitude and direction by AB, AD, and they act at O. Hence, by the Parallelogram of Forces, their resultant is similarly represented by AC, and also acts at O.

But Q is represented by CA.

Therefore the resultant of R, P is equal and opposite to the third force Q.

Therefore the three forces are in equilibrium. [Q.E.D.]

OBSERVATIONS.—The three forces *must not act along the sides of the triangle.* They must act *at a point* in directions parallel to these sides, as in Fig. 5; otherwise they cannot be applied to the same particle, and the proof fails.

The angle between any two forces is the **supplement** *of the corresponding interior angle of the triangle.* Thus

$\angle$ between R, P (*i.e.*, $\angle ROP$ in Fig. 5) $= \angle BAD = 180^\circ - \angle ABC$.

* The sides of a triangle or polygon are said to be *taken in order* when of any two adjacent sides one is drawn towards and the other away from their common angular point.

17. Converse of the Triangle of Forces.

If three forces acting on a particle are in equilibrium, any triangle whose sides are parallel to the directions of the forces shall have the lengths of these sides proportional to the magnitudes of the forces.

Let P, Q, R be three forces in equilibrium acting at O, and let ABC be any triangle whose sides BC, CA, AB are parallel to P, Q, R.

Then, if the scale of representation be properly chosen, BC, CA, AB shall represent P, Q, R in magnitude as well as direction.

For let the length BC be chosen to represent P. (§ 10.)

If CA does not represent Q, let CK represent Q.

Then, by the Parallelogram or Triangle of Forces, the resultant of P and Q is represented by BK. But P, Q, R

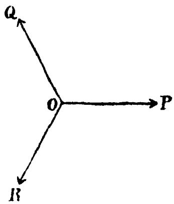


Fig. 6.

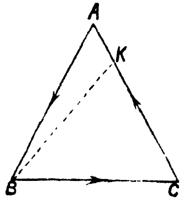


Fig. 7.

are in equilibrium. Hence R must be represented by the line KB, equal and opposite to the resultant BK. But, by hypothesis, R acts in the direction AB. Therefore Q cannot be represented in magnitude by any other length than CA, and therefore also, by the above reasoning, R is represented by AB.

Observation. — The above condition may be expressed by the relations $\frac{P}{Q} = \frac{BC}{CA}$, $\frac{Q}{R} = \frac{CA}{AB}$, $\frac{R}{P} = \frac{AB}{BC}$, which may also be written

$$\frac{P}{BC} = \frac{Q}{CA} = \frac{R}{AB}.$$

It is proved in Euclid, Book VI., Prop. 4, that any triangle whose angles are equal to those of ABC has its sides proportional to those of ABC. Hence, if any triangle be drawn whose sides are parallel to BC, CA, AB, these sides will also represent P, Q, R, but on a different scale.

18. The Polygon of Forces.—*If any number of forces acting on the same particle can be represented in magnitude and direction by the sides of a closed polygon taken in order, they shall be in equilibrium.*

Let the forces P, Q, R, S, acting on a particle at O, be represented in magnitude and direction (but not in

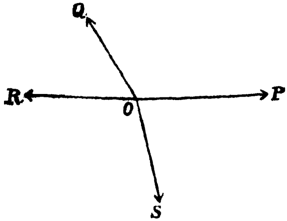


Fig. 8.

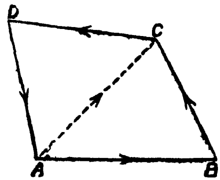


Fig. 9.

position) by the sides AB, BC, CD, DA of a polygon $ABCD$.

Then shall the forces be in equilibrium.

For, as in the Triangle of Forces, or § 15, the resultant of the forces P, Q is represented in magnitude and direction by AC.

Therefore the resultant of P, Q, R is the same in magnitude and direction as that of forces AC and CD, and therefore similarly represented by AD.

But the force S is represented by DA.

Therefore S is equal and opposite to the resultant of P, Q, R.

Therefore the forces P, Q, R, S are in equilibrium. [Q.E.D.]

The observations at the end of § 16 are equally applicable to the Polygon of Forces.

We have considered the case of four forces, but the proof may be similarly extended to any number of forces.

19. **To construct the resultant of any number of forces acting on a particle.**

Let the given forces be represented by the straight lines AB, BC, CD, taken in order, forming all the sides but one of a polygon. Then, if the polygon be completed by drawing the remaining side *from* A, the extremity of the

first side, *to* D, the extremity of the last side, the line AD will represent the resultant force.

This is evident from the last article.

It is immaterial in what succession the forces are represented. The form of the polygon will depend on which force is represented first, which next, and so on; but the line representing the resultant will be the same in every case.

For, consider the case of two forces P, Q, acting at A (Fig. 1). If we represent P first and Q second, the lines representing them will be AB, BC, respectively, and the resultant will be represented by AC. If we represent Q first and P second, the lines representing them will be the opposite sides AD, DC, respectively, and therefore the resultant will still be represented by AC. The same property may be extended to any number of forces by interchanging their order of succession of representation, taking two at a time.

20. Converse of the Polygon of Forces.

If any number of forces acting on a particle are in equilibrium, a closed polygon can be drawn whose sides represent these forces both in magnitude and direction.

Let the forces P, Q, R, S, acting at O, be represented by AB, BC, CD, DE, respectively, these lines being joined

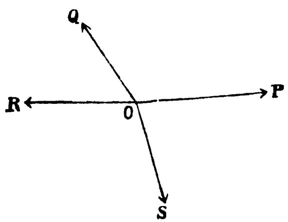


Fig. 10.

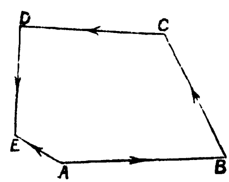


Fig. 11.

end to end. Then, if the figure $ABCDE$ is not a closed polygon, the forces will have a resultant represented by AE (§ 19), and will, therefore, not be in equilibrium.

Therefore the lines representing the forces must be capable of being formed into a closed polygon.

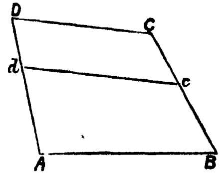


Fig. 12.

21. *The converse of the Polygon of Forces is less complete than that of the Triangle.*—For, if there are more than three orces, we can draw any number of different polygons, such as $ABCD$, $ABcd$ (Fig. 12), each having its sides parallel to the directions of the forces. The sides of each polygon represent *a* system of forces in equilibrium acting in these directions, but this is not necessarily *the* system considered. Hence there are any number of such systems all different. For example, a particle could be kept in equilibrium by three forces acting in any three of the given directions only. We cannot, therefore, except in the case of three forces, determine the ratios of the forces from knowing their directions.

22. Applications of the Parallelogram, Triangle, and Polygon of Forces.

(1) *The resultant of two equal forces bisects the angle between them.*

(2) *If any three forces are in equilibrium, any two of the forces are together not less than the third.* For, in the Triangle of Forces, any two sides are together greater than the third (Euc. I. 20).

If two forces are together *equal* to the third, they will balance if the first two forces act in the same straight line and in the opposite sense to the third.

(3) *The resultant of two forces P, Q is greatest* when both act in the same direction, and is then $P+Q$. It is *least* when they act in opposite senses in the same straight line, and is then either $P-Q$ acting in the direction of P or $Q-P$ in the direction of Q.

[This is obvious from (2).]

(4) *If three equal forces are in equilibrium, the angle between any two of them is* 120°. For the Triangle of Forces is equilateral; therefore each of its angles is 60°, and the angle between the corresponding pair of forces

$$= 180° - 60° = 120°.$$

(5) *The Perpendicular Triangle of Forces.*—If three forces proportional to the sides of a triangle act on a particle in directions perpendicular to these sides taken in order, they shall be in equilibrium.

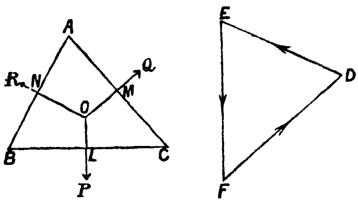


Fig. 13.

For if the triangle *ABC* be turned through a right angle into the position *DEF*, its sides, taken in order, will be brought parallel to the forces, and will therefore represent the forces both in magnitude and direction. Therefore they will be in equilibrium.

[The student may satisfy himself of the legitimacy of this proof by cutting out a triangle in paper and placing it in the two positions *ABC*, *DEF* in succession.]

(6) *The Perpendicular Polygon.* — Generally, if any number of forces proportional to the sides of a closed polygon act on a particle in directions perpendicular to these sides taken in order, they shall be in equilibrium. For, on turning the polygon through a right angle, it will become the Polygon of Forces.

(7) *If two forces be represented by the sides of a triangle **both drawn from** their point of application, their resultant is represented by twice the bisector of the base drawn from that point.*

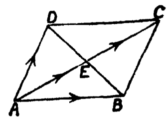


Fig. 14.

For let *AB*, *AD* represent the forces. Complete the parallelogram of forces *ABCD*; then *AC* represents the resultant. But the diagonals of a parallelogram bisect each other. Hence *AC* bisects *BD* in *E*, and *AC* is twice the length of the bisector *AE*.

(8) *To find where a particle O must be placed inside a triangle ABC so that it may be in equilibrium under forces to the vertices represented by OA, OB, OC.*

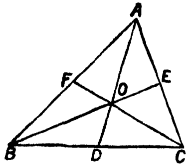


Fig. 15.

Let D be the middle point of BC. Then, by the last proposition, the forces OB, OC have a resultant $2OD$ along OD. This must be equal and opposite to the third force OA. Hence O must lie on the bisector AD at a point such that $OA = 2DO$.

$$\therefore \quad DA = 3DO \quad \text{and} \quad DO = \tfrac{1}{3}DA.$$

23. *Example.* — Two forces act along the sides AB, AC of a triangle, and are represented in magnitude by AB and three times AC respectively. To find where their resultant cuts the base, and to determine its magnitude.

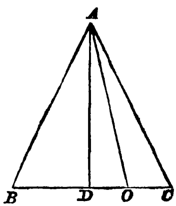


Fig. 16.

Let the resultant cut the base in O. Then, by the Triangle of Forces, the force AB is equivalent to forces AO along AO, OB at A parallel to OB.

Similarly the force $3AC$ along AC is equivalent to forces $3AO$ along AO, $3OC$ at A parallel to OC.

Therefore the two given forces are equivalent to forces of $4AO$ along AO, OB and $3OC$ in opposite directions parallel to BC.

If the resultant acts along AO, the two latter forces must balance.

$$\therefore \quad OA = 3OC, \quad \text{and} \quad \therefore \quad BC = 4OC;$$

$$\therefore \quad OC = \tfrac{1}{4}BC, \quad \text{and} \quad BO = \tfrac{3}{4}BC.$$

Also the resultant is the remaining force, viz. $4AO$ acting along AO.

24. **Two forces act along two sides of a triangle, and their magnitudes are given multiples of these sides. To find their resultant.**

Case 1. Let the forces be $m \,.\, AB$ acting along AB, and $n \,.\, AC$ along AC (Fig. 17).

Let their resultant cut the base BC in O.

Replace each force by two component forces along AO, AE and parallel to BC.

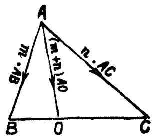


Fig. 17.

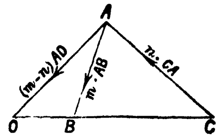


Fig. 18.

Then, by the Triangle of Forces, the force $m \, . \, AB$ is equivalent to forces $m \, . \, AO$ along AO,

$m \, . \, OB$ at A parallel to OB.

Similarly, the force $n \, . \, AC$ is equivalent to forces

$n \, . \, AO$ along AO,

$n \, . \, OC$ at A parallel to OC.

Hence the two forces together are equivalent to

$(m+n) \, AO$ along AO,

$n \, . \, OC - m \, . \, BO$ at A parallel to BC.

But, if the resultant acts along AO, the latter component must vanish.

$$\therefore \quad n \, . \, OC = m \, . \, BO \, ;$$

$$\therefore \quad (m+n) \, OC = m \, (BO+OC) = m \, . \, BC,$$

$$(m+n) \, BO = n \, (BO+OC) = n \, . \, BC \, ;$$

$$\therefore \quad BO = \frac{n}{m+n} BC, \qquad OC = \frac{m}{m+n} BC \, ;$$

which determine O.

Also, the resultant is the remaining force, $(m+n) \, AO$ acting along AO.

CASE 2. Let the forces be $m \, . \, AB$ acting along AB, and $n \, . \, CA$ along CA (in the *opposite* sense to AC, Fig. 18).

If $m = n$, the resultant is a force $m \, . \, CB$ parallel to CB (§§ 15, 16).

If not, let the resultant cut CB produced in O.

The components of the force $m \, . \, AB$ are $m \, . \, AO$ and $m \, . \, OB$ as before, but those of $n \, . \, CA$ are represented in

magnitude, direction, and sense by $n \cdot CO$ and $n \cdot OA$, respectively. Hence the two forces are now equivalent to forces

$$(m-n)\,AO \text{ along } AO,$$
$$m \cdot OB - n \cdot OC \text{ at } A \text{ parallel to } BC;$$

$\therefore$ as before $m \cdot OB = n \cdot OC$;

$$\therefore\ (m-n)\,OB = n\,(OC-OB) = n \cdot BC,$$
$$(m-n)\,OC = m\,(OC-OB) = m \cdot BC;$$
$$\therefore\ OB = \frac{n}{m-n}BC, \quad OC = \frac{m}{m-n}BC;$$

and the resultant is $(m-n)\,AO$ along AO.

[This case might be deduced from Case 1 by writing $-n$ for n.]

Example.—ABC is a triangle, and E the middle point of BC. To find the resultant of forces at A represented in magnitude, direction, and sense by $2BA$, $2AE$, and AC respectively.

By the Triangle of Forces, the resultant of forces $2BA$, $2AE$ is a force $2BE$ parallel to BC. But $2BE = BC$; therefore the three forces are equivalent to forces BC, AC acting at A.

Produce BC to F, so that $CF = BC$.

Then the two last forces are represented by CF, AC respectively, and they act at A.

Therefore the required resultant is represented by AF.

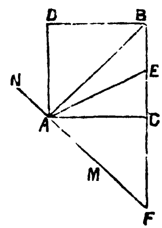


Fig. 19.

25. To find the magnitude of the resultant of two forces P, Q in directions at right angles to one another.

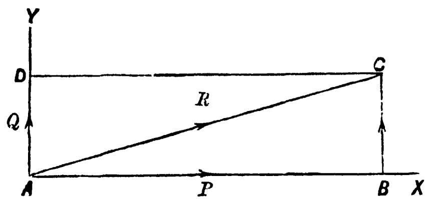


Fig. 20.

Let AB, AD represent the two forces P, Q. Complete the rectangle $ABCD$

Then AC represents the resultant force.

Let $AC = R$. Then, by Euclid I. 47,

$$AC^2 = AB^2 + BC^2 = AB^2 + AD^2;$$

$$\therefore \quad R^2 = P^2 + Q^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots \quad (1);$$

$\therefore$ **resultant force** $R = \sqrt{(P^2 + Q^2)}$.

Example.—To find the resultant of three forces of 1 lb. acting at a point, the angle between the first and second being 90°, and that between the second and third being 45°.

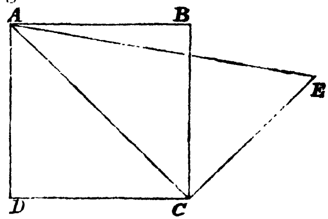


Fig 21.

Draw AD, DC, CE, each of unit length, parallel to the forces. Then AC represents the resultant of the two first forces, and AE that of the three.

Now

$$AC^2 = AD^2 + DC^2 = 1^2 + 1^2 = 2.$$

Also AC is evidently the diagonal of the square $ABCD$;

$$\therefore \quad \angle ACD = 45°.$$

But, by § 17, Observation,

$$\angle DCE = \text{supplement of angle between forces} = 180° - 45° = 135°;$$

$$\therefore \quad ACE = 90°;$$

$$\therefore \quad AE^2 = AC^2 + CE^2 = 2 + 1^2 = 3, \quad i.e., \ AE = \sqrt{3} = 1{\cdot}732\ldots.$$

Therefore the resultant = 1·732 lbs., approximately.

On measuring $\angle BAE$ with a protractor, it will be found to be about 10°. Hence the resultant makes an angle of about 10° with the middle force.

[As an exercise the student should draw the diagram on a large scale, and by actual measurement verify that AE is approximately 1·732 times AB.]

26. **Application to velocities and accelerations.**—Since forces on a particle are compounded by the same law as component velocities and accelerations, it follows that *all* the properties proved, both in this and the next chapter, hold equally good for component and resultant velocities or accelerations as well as for forces.

Summary of Results.

A straight line can represent a force (i.) in point of application, (ii.) in direction and sense, (iii.) in magnitude. (§ 9.)

The Parallelogram of Forces.—If two forces acting on the same particle be represented by two adjacent sides of

a parallelogram drawn from their point of application, their resultant shall be represented by the diagonal of the parallelogram drawn from that point. (§ 12.)

The Triangle and Polygon of Forces.—If three or more forces acting on the same particle can be represented in magnitude and direction (but not in position) by the sides of a triangle or closed polygon taken in order, they shall be in equilibrium. (§§ 16, 18.)

Converse of "Triangle."—If three forces (P, Q, R) acting on a particle balance, *any triangle* (ABC), whose sides are parallel to the forces, shall have these sides proportional to their magnitudes (§ 17), or

$$\frac{P}{BC} = \frac{Q}{CA} = \frac{R}{AB}.$$

Converse of "Polygon." — If *more* than three forces balance, *a polygon can be drawn* whose sides represent them in magnitude and direction. (§§ 20, 21.)

Resultant of forces $m \,.\, AB$ and $n \,.\, AC$ along AB, AC is force $(m+n)\, AO$ along AO, where $m \,.\, BO = n \,.\, OC$. (§24.)

Resultant of two perpendicular forces P, Q is R, where

$$R^2 = P^2 + Q^2 \; \ldots\ldots\ldots\ldots \; (1). \quad (\S\ 25.)$$

EXAMPLES I.

1. Draw a diagram, as well as you can to scale, showing the resultant of two forces, equal to the weights of 6 and 12 lbs., acting on a particle, with an angle of 60° between them; and, by measuring the resultant, find its numerical value.

2. The greatest resultant which three given forces acting at a point can have is 30 lbs., and the least is 2 lbs. What is the magnitude of the greatest force?

3. The greatest resultant which three given forces acting at a point can have is 30 lbs., and the least is 0. Find the limits between which the greatest force must lie.

4. ABC is a triangle. Find the resultant of forces at A represented in magnitude and direction and sense by

(i.) $3AB$ and $5AC$;	(iii.) BA and $3AC$;	(v.) $4AB$, $5AC$, $6BC$;
(ii.) $2BA$ and $3CA$;	(iv.) $4AB$ and $5CA$;	(vi.) $3AB$, $4CA$, $5CB$.

5. A body is pulled north, south, east, and west by four strings whose directions meet in a point, and the forces of tension in the strings are equal to 10, 15, 20, and 32 lbs. weight respectively. What is the magnitude of the resultant?

6. Forces P, $2P$, $3P$ act at a point in directions parallel to the sides of an equilateral triangle taken in order. Find their resultant.

7. Show how to find, graphically or otherwise, the resultant of a number of forces acting on a rigid body at one point, and apply your method to find the resultant of forces 1 in an easterly direction, $\sqrt{2}$ in a north-easterly direction, and 1 to the north.

8. What angle must two forces of 5 and 12 lbs. include if they are balanced by a force of 13 lbs.?

9. Show that the resultant of two perpendicular forces $P+Q$ and $P-Q$ is equal in magnitude to the resultant of two perpendicular forces $\sqrt{2}P$ and $\sqrt{2}Q$.

10. The sides BC, CA, AB of a triangle are bisected in D, E, F, respectively. Find the resultant of forces represented by DA, EB, FC.

11. ABC is any triangle. At A are applied forces $k.BC$, $l.CA$, $m.AB$, parallel to BC and along CA, AB, respectively. Show that their resultant is the resultant of forces $(k-l)AC$ along AC, and $(m-k)AB$ along AB, and hence find where it cuts the base BC.

12. $ABDC$ is a parallelogram; E is a point in AC. Find a point F in BD such that the resultant of forces represented by AE and AF may act in the direction AD.

13. If A, B, C are any three points in a straight line, O any point not in that straight line; if a force represented in magnitude and direction by OA act from O to A; if a force represented in magnitude and direction by BO act from B to O, and a force represented in magnitude and direction by CO act from C to O; then show that the resultant of these three forces cuts the straight line ABC in a point D such that $AB = CD$.

14. Forces P, Q, whose resultant is R, act at a point O, and a line is drawn meeting their directions in L, M, N; show that

$$P/OL + Q/OM = R/ON.$$

CHAPTER II.

RECTANGULAR RESOLUTION OF FORCES.

27. **Resolution of Forces.**—The Parallelogram of Forces tells us that any two forces acting on a particle are equivalent to a single resultant force represented by the diagonal of the parallelogram whose sides represent the forces themselves. Conversely, if a single force be represented by a straight line, and we draw any parallelogram having this line for a diagonal, the given force may be replaced by two forces represented by the sides of the parallelogram.

In like manner, the Polygon of Forces (or rather the construction of § 19) tells us that a force represented by one side of a polygon may be replaced by a number of forces having the same point of application, and represented in magnitude and direction by the remaining sides of the polygon taken the other way round.

DEFINITIONS.—The process of replacing a single force by two or more forces having that force for their resultant is known as the **resolution** of forces, and is the reverse process to the *composition* of forces.

The several forces are called the **components** of the given force.

Thus we speak of *resolving* a force into *components*, and of *compounding* two or more forces into a single *resultant*.

To resolve a force into components in two given directions AB, AD, it is only necessary to draw the straight line AC representing the given force, and to draw CD, CB through C parallel to BA, DA, respectively. Then, by the Parallelogram of Forces, AB, AD will represent the required components.

28. A force P is inclined at an angle A to a given line. To resolve P into components along and perpendicular to that line.

Let AC represent the given force P, and AB be the given line, so that $\angle BAC = A$.

Draw AD perpendicular to AB, and complete the parallelogram $ABCD$. Then AB, AD represent the two required components. Let X, Y denote these components respectively.

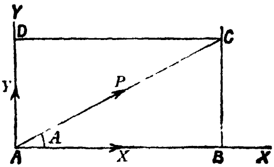


Fig. 22.

Since BC is perpendicular to AB, therefore, by definition (Trig., § 4),

$$\therefore \quad \cos BAC = \frac{AB}{AC} \quad \text{and} \quad \sin BAC = \frac{BC}{AC}.$$

$$\therefore \quad AB = BC \cos BAC = AC \cos A$$

and $$AD = AC \sin BAC = AC \sin A.$$

$$\therefore \quad \mathbf{X = P \cos A} \quad \text{and} \quad \mathbf{Y = P \sin A} \ \ldots\ldots\mathbf{(1)}.$$

Cor. The student will have no difficulty in verifying the following important results:—*

Where the angle A =	30°	45°	60°
The component X =	$\frac{\sqrt{3}}{2} . P$	$\sqrt{\frac{1}{2}} . P$	$\frac{1}{2}P$
The component Y =	$\frac{1}{2}P$	$\sqrt{\frac{1}{2}} . P$	$\frac{\sqrt{3}}{2} . P$

Example.—(1) A force of 10 lbs. makes an angle of 135° with a given line. To resolve it into components along and perpendicular to that line.

Let $XAC = 135°$, and let AC represent 10 lbs.

Produce XA to B, and complete the rectangle $ABCD$.

Then AB, AD represent the required components.

Now $\angle BAC = 180° - 35° = 45°$.

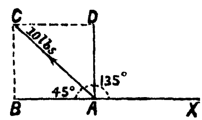


Fig. 23.

* These results should either be remembered, or the student should be able to obtain them instantly.

Hence the components along AB and AD are

$$AB = AC\sqrt{\tfrac{1}{2}} = 10 \times \tfrac{1}{2}\sqrt{2} = 5\sqrt{2} \text{ lbs.},$$

$$AD = AC\sqrt{\tfrac{1}{2}} = 10 \times \tfrac{1}{2}\sqrt{2} = 5\sqrt{2} \text{ lbs.}$$

But the force $5\sqrt{2}$ lbs. along AB acts in the *opposite* direction to that in which AX is drawn; hence this force is to be regarded as a *minus quantity*.

Therefore the required components are

$$-5\sqrt{2} \text{ lbs. along } AX, \qquad +5\sqrt{2} \text{ lbs. perpendicular to } AX.$$

Alternative Method.—Or, by the formula (1) and Trig. § 18, we have at once

$$X = 10\cos 135^\circ = 10 \times -\tfrac{1}{2}\sqrt{2} = -5\sqrt{2} \text{ lbs.},$$

$$Y = 10\sin 135^\circ = 10 \times \tfrac{1}{2}\sqrt{2} = +5\sqrt{2} \text{ lbs.}$$

(2) Three forces of 5 lbs., 6 lbs., and 4 lbs. are inclined to one another at angles of 120°. To replace them by two forces acting along and perpendicular to the force of 5 lbs., and to find the resultant of the three.

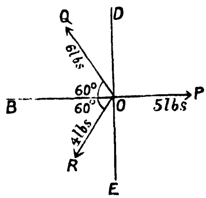


Fig. 24.

Let OP, OQ, OR represent the three forces. Produce PO to B, and draw DOE perpendicular to OP.

Then $BOQ = BOR = 60^\circ$.

Therefore the force 6 lbs. along OQ is equivalent to

$$6 \times \tfrac{1}{2} \text{ lbs. along } OB$$

and $\quad 6 \times \tfrac{1}{2}\sqrt{3}$ lbs. along OD,

or to $\quad -3$ lbs. along OP

and $\quad 3\sqrt{3}$ lbs. along OD.

Similarly, the force 4 lbs. along OR is equivalent to

$$4 \times \tfrac{1}{2} \text{ lbs. along } OB \quad \text{and} \quad 4 \times \tfrac{1}{2}\sqrt{3} \text{ lbs. along } OE,$$

or to $\quad -2$ lbs. along OP and $-2\sqrt{3}$ lbs. along OD.

We also have the force 5 lbs. along OP.

Therefore the three forces are equivalent to

$$5-3-2 \text{ lbs. along } OP \quad \text{and} \quad 3\sqrt{3}-2\sqrt{3} \text{ lbs. along } OD,$$

i.e., $\quad$ zero along OP and $\sqrt{3}$ lbs. along OD.

Therefore the required resultant acts along OD, and its magnitude

$$= \sqrt{3} \text{ lbs.} = 1\cdot732 \text{ lbs. nearly.}$$

29. Definition.—**The resolved part or resolute of a force** along a given straight line is the component of the force along that line *when the other component is perpendicular to that line.*

Thus, let AB be the given straight line, and let the given force P be represented by AC. Drop CM perpendicular on AB.

Then AM, MC represent the components of P along and perpendicular to AB.

Therefore AM represents the resolved part of P along AB.

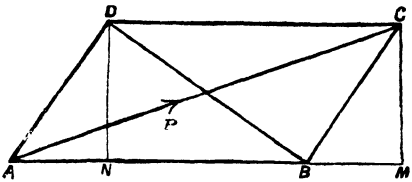


Fig. 25.

If a force is resolved into two components in directions which are *not at right angles*, these components are *not* the resolved parts of the force in these directions.

For if any parallelogram $ABCD$ be drawn, having AC as diagonal, AB, AD represent component forces, having P for their resultant, but AB does *not* represent the resolved part of P along AM.

Since $$AM = AC \cos MAC,$$

$\therefore$ **resolved part of P along $AB = P \cdot \cos BAC$,**

or *the resolved part of a force in a given direction is equal to the product of the force into the cosine of the angle which it makes with the given direction.*

For the angles 30°, 45°, 60° the resolved parts are therefore $\frac{1}{2}\sqrt{3}\,P$, $\frac{1}{2}\sqrt{2}\,P$, $\frac{1}{2}P$, respectively (§ 28, Cor.).

If the angle is obtuse, the resolved part is considered negative, as in § 28, Exs. 1, 2.

Example 2 of § 28 shows that the resultant of any number of forces may sometimes best be found by first resolving each force into two components at right angles. We shall now consider this method generally.

30. To find the resultant of any number of forces acting on a particle in one plane.

Let P_1, P_2, P_3 ... denote several forces acting at A.

Take any direction AX in the plane of the forces, and draw AY perpendicular to AX. Resolve each force into two components in the directions AX, AY (§ 28, Fig. 22).

Let the components of P_1 be X_1, Y_1, respectively,
" " P_2 " X_2, Y_2, "
and so on; these components being taken with the sign + or −, according to the direction in which they act.

Thus the system of forces is equivalent to forces X_1, X_2, X_3 ... along OX, and Y_1, Y_2, Y_3 ... along OY.

Now these forces may be compounded in any order (§ 19).

Let us first compound together all the forces acting along OX. Then, if X denote their resultant, we have

$$X = X_1 + X_2 + X_3 + \dots .$$

Let us next compound together all the forces acting along OY. Then, if Y denote their resultant, we have

$$Y = Y_1 + Y_2 + Y_3 + \dots .$$

The whole system of forces is therefore reduced to two known forces—X along OX, and Y along OY. Hence, if R denote the final resultant of all the forces, we have, by § 25,

$$\boldsymbol{R^2 = X^2 + Y^2 = (X_1 + X_2 + \dots)^2 + (Y_1 + Y_2 + \dots)^2} \dots \mathbf{(2)}.$$

Observation.—The line OX may be drawn in any convenient direction, but it is always advisable (and in fact necessary in all elementary problems) to draw this line in such a direction that the given forces make angles of 0°, 30°, 45°, 60°, or 90° with it or with its direction produced backwards through O. It is best, when possible, to take OX in the direction of one of the component forces.

Example.—(1) To find the resultant of two forces of 1 lb. and 3 lbs. inclined at an angle 30°.

Resolve the second force into components along and perpendicular to the first. These components are $3 \times \frac{1}{2}\sqrt{3}$ and $3 \times \frac{1}{2}$ lbs. respectively.

Hence the two forces are together equivalent to forces $1 + \frac{3}{2}\sqrt{3}$ lbs. and $\frac{3}{2}$ lbs. acting along and perpendicular to the first force.

Let their resultant be R. Then, by § 25,

$$R^2 = (1 + \tfrac{3}{2}\sqrt{3})^2 + (\tfrac{3}{2})^2 = 1 + 3\sqrt{3} + \tfrac{27}{4} + \tfrac{9}{4} = 10 + 3\sqrt{3};$$

$$\therefore \quad R = \sqrt{(10 + 3\sqrt{3})} = \sqrt{(10 + 3 \times 1{\cdot}732)} = \sqrt{15{\cdot}196};$$

$$\therefore \quad \text{the resultant} = 3{\cdot}898 \text{ lbs. nearly.}$$

31. To find the magnitude of the resultant of two forces inclined to one another at any angle.— We shall now show that

Square of resultant of two forces = sum of squares of components + twice product of one force into resolved part of the other force along its line of action.

Let AB, AD represent the components P, Q.

Complete the parallelogram $ABCD$. Then AC represents the resultant R.

Drop CM, DN perpendicular on AB.

Then $BM = AN = X$, the resolved part of Q along AB.

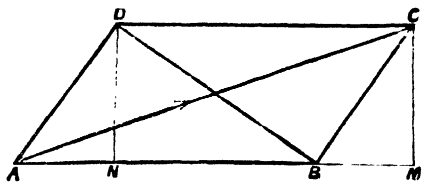


Fig. 26.

By Euc. II. 12, $AC^2 = AB^2 + BC^2 + 2AB \cdot BM$.

Therefore $\qquad R^2 = P^2 + Q^2 + 2PX$ .... **(3)**.

If $\angle BAD$ is obtuse, X is negative, and $\angle ABC$ is acute, and the same thing follows from Euc. II. 13.

Cor. 1. By § 28 we have $X = Q \cos$ (angle between P and Q), which for brevity we shall write $= Q \cos (P, Q)$.

Therefore the above relation may be written

$$R^2 = P^2 + Q^2 + 2PQ \cos (P, Q) \text{ ....} (3a).$$

Observe that the same formulæ are also applicable to velocities and accelerations.

Cor. 2.

If the $\angle$ between the forces is	0°	30°	45°	60°	90°	120°	135°	150°	180°
then $R^2 =$ $P^2 + Q^2$ + $2PQ$ times	$+\frac{\sqrt{4}}{2}$	$+\frac{\sqrt{3}}{2}$	$+\frac{\sqrt{2}}{2}$	$+\frac{\sqrt{1}}{2}$	$\frac{\sqrt{0}}{2}$	$-\frac{\sqrt{1}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{4}}{2}$

We can now write down these results in a form which can be easily remembered:—*

For	$0°$,	$R^2 = P^2 + Q^2 + \sqrt{4} \,.\, PQ$;
	$30°$,	$R^2 = P^2 + Q^2 + \sqrt{3} \,.\, PQ$;
	$45°$,	$R^2 = P^2 + Q^2 + \sqrt{2} \,.\, PQ$;
	$60°$,	$R^2 = P^2 + Q^2 + \sqrt{1} \,.\, PQ$;
	$90°$,	$R^2 = P^2 + Q^2 + \sqrt{0} \,.\, PQ$;
	$120°$,	$R^2 = P^2 + Q^2 - \sqrt{1} \,.\, PQ$;
	$135°$,	$R^2 = P^2 + Q^2 - \sqrt{2} \,.\, PQ$;
	$150°$,	$R^2 = P^2 + Q^2 - \sqrt{3} \,.\, PQ$;
	$180°$,	$R^2 = P^2 + Q^2 - \sqrt{4} \,.\, PQ$.

Examples.—(1) To find the resultant of forces of 7 lbs. and 11 lbs. inclined at an angle of 60°.

The resolved part of the second force in the direction of the first $= 11 \cos 60° = 5\frac{1}{2}$ lbs.

Therefore, if the resultant contains R lbs.,

$$R^2 = 7^2 + 11^2 + 2 \,.\, 7 \,.\, 5\tfrac{1}{2}$$
$$= 49 + 121 + 77 = 247,$$

whence $\quad R = \sqrt{247} = 15{\cdot}716$ lbs. wt. (*cf.* § 14, Ex.).

(2) To find the resultant when the same forces include an angle of 120°.

If the direction of the 7 lb. force is produced backwards, the 11 lb. force will be found to make an angle 60° with it. Hence the resolved part of the latter force is $5\frac{1}{2}$ lbs. in the reverse direction to the 7 lb. force. It must therefore be considered negative and called $-5\frac{1}{2}$ lbs.

Hence $\quad R^2 = 7^2 + 11^2 + 2 \,.\, 7 \,.\, (-5\tfrac{1}{2})$

$$= 49 + 121 - 77 = 93,$$

whence $\quad R = \sqrt{93} = 9{\cdot}643$ lbs. wt.

---

* These results may be committed to memory, but we recommend the student to deduce them from the formula $R^2 = P^2 + Q^2 + 2PX$. See appendix on *Trigonometry*.

32. *To find the resultant of two forces P, Q, when the angle between their directions is any multiple of* 15°.

In § 31, Cor. 2, the expressions for the resultant are given for all angles that are multiples of 15°, with the exception of 15°, 75°, 105°, 165°.

If the angle between the forces has any one of these four values, we must draw two perpendicular lines OX, OY inclined at angles of 45° to the direction of one of the forces, say P. Then it will be seen from Figs. 27–30 that the other force Q makes angles of 30° and 60° with the lines OX, OY, or these lines produced. Hence the forces P, Q can be replaced by their components along OX, OY, as in § 28, Cor. 1, and their resultant may be found as in § 30.

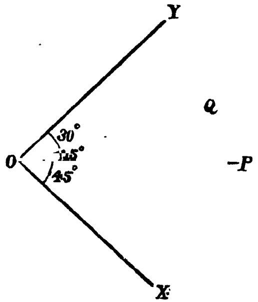


Fig. 27.
Angle between forces = 15°.

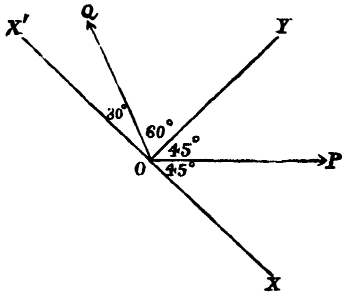


Fig. 28
Angle between forces = 75°.

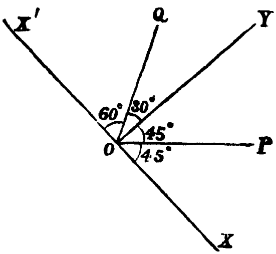


Fig. 29.
Angle between forces = 105°.

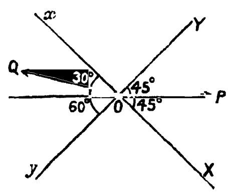


Fig. 30.
Angle between forces = 165°.

*Example.**—To find the resultant of two forces of 4 lbs. and 2 lbs. inclined at an angle of 165°.

Let OP, OQ (Fig. 30) represent the directions of the forces of 2 lbs. and 4 lbs.

Draw XOx making an angle 45° with OP, and draw YOy perpendicular to XOx.

Then $\angle xOQ = POQ - POY - YOx = 165° - 45° - 90° = 30°$,

$\angle QOy = 60°$.

Therefore the components of the force of 4 lbs. acting along OP are

$4 \times \sqrt{\tfrac{1}{2}}$ or $2\sqrt{2}$ lbs. along OX,

$4 \times \sqrt{\tfrac{1}{2}}$ or $2\sqrt{2}$ lbs. along OY;

and the components of the force of 2 lbs. acting along OQ are

$2 \times \tfrac{1}{2}\sqrt{3}$ or $\sqrt{3}$ lbs. along Ox,

$2 \times \tfrac{1}{2}$ or 1 lb. along Oy.

Therefore the given forces are equivalent to forces X, Y along OX, OY, where

$$X = 2\sqrt{2} - \sqrt{3},$$
$$Y = 2\sqrt{2} - 1.$$

Therefore, if R be the required resultant,

$$\begin{aligned} R^2 &= X^2 + Y^2 = (2\sqrt{2} - \sqrt{3})^2 + (2\sqrt{2} - 1)^2 \\ &= 8 - 4\sqrt{6} + 3 + 8 - 4\sqrt{2} + 1 \\ &= 20 - 4(\sqrt{6} + \sqrt{2}) = 4(5 - \sqrt{6} - \sqrt{2}); \\ \therefore\ R &= 2\sqrt{(5 - \sqrt{6} - \sqrt{2})} \text{ lbs.} \\ &= 2{\cdot}13 \text{ lbs. approximately (by calculation).} \end{aligned}$$

33. **Conditions of equilibrium.**—In order that a system of forces acting on a particle in one plane should be in equilibrium, it is necessary and sufficient that the sums of the resolved parts of the forces along two straight lines at right angles should be separately zero.

Let OX, OY be two straight lines at right angles. Then, by § 30, we may replace the given forces by two forces X, Y, acting along OX, OY, such that

$X = X_1 + X_2 + \ldots =$ sum of resolved parts of forces along OX,

$Y = Y_1 + Y_2 + \ldots =$,, ,, ,, ,, OY;

and the resultant R is given by

$$R^2 = X^2 + Y^2.$$

* For examples of the other three cases, see *Worked Examples in Mechanics*, pages 83, 84.

Now two forces cannot be in equilibrium unless they act in the same straight line. Hence, for equilibrium, we must have

$$0 = X = \text{sum of resolved parts along } OX,$$
$$0 = Y = \quad ,, \quad ,, \quad ,, \quad ,, \; OY.$$

Conversely, if these two conditions are satisfied, $R = 0$, and the forces are in equilibrium.

OBSERVATIONS.—If X were zero and Y were not zero, the forces would have a resultant Y perpendicular to OX.

The proposition shows that, if the forces are in equilibrium, the sum of the resolved parts along *every* straight line is zero. But this will *necessarily* be the case if the sums of their resolved parts along *two* perpendicular straight lines is zero. The same thing is true if the two straight lines are not perpendicular. For, the forces were not in equilibrium, their resultant would have to be perpendicular to both lines, which is impossible.

Example. — To show that three forces of $\sqrt{2}$ lbs., 2 lbs., and $(\sqrt{3}-1)$ lbs. acting on a particle are in equilibrium if the second and third include an angle of 150°, and the third and first an angle of 45°.

Let OP, OQ, OR be the lines of action of the forces.

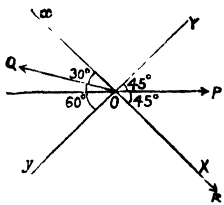


Fig. 31.

Take any point X on OR. Produce XO to x, and draw YOy perpendicular to it.

Then the components of the force of $\sqrt{2}$ lbs. along OP are

1 lb. along OX and 1 lb. along OY.

The components of the force of 2 lbs. along OQ are

$\sqrt{3}$ lbs. along Ox and 1 lb. along Oy;

and we have also a force of $\sqrt{3}-1$ lbs. along OX;

$\therefore$ the three forces are equivalent to forces of

$$1-\sqrt{3}+\sqrt{3}-1 \text{ lbs. along } OX, \quad 1-1 \text{ lbs. along } OY.$$

But these are each zero.

Therefore the forces are in equilibrium.

***34. Lami's Theorem.**—*If three forces keep a particle in equilibrium, each force is proportional to the sine of the angle between the other two.*

Let the forces P, Q, R act along OP, OQ, OR. If they are in equilibrium, we have, by § 32,

sum of resolved parts perpendicular to $OR = 0$.

But R has no resolved part perpendicular to OR.

Therefore resolved parts of P, Q perpendicular to OR are equal and opposite.

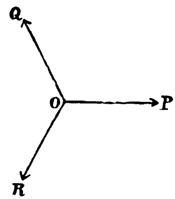


Fig. 32.

$$\therefore \quad P \sin ROP = Q \sin QOR.$$

$$\therefore \quad \frac{P}{Q} = \frac{\sin QOR}{\sin ROP}.$$

Similarly, by resolving perpendicular to OP, we have

$$\frac{Q}{R} = \frac{\sin ROP}{\sin POQ}.$$

$$\therefore \quad \frac{P}{\sin QOR} = \frac{Q}{\sin ROP} = \frac{R}{\sin POQ} \quad \cdots\cdots\cdots \quad \textbf{(4)};$$

as was to be proved.

[This relation can also be deduced from the Triangle of Forces, by means of a well-known theorem in Trigonometry which asserts that in any triangle each side is proportional to the sine of the angle between the other two.]

35. **To find the direction of the resultant of two given forces X, Y acting at right angles.**

If the forces X, Y are represented by AB, AD, their resultant is represented by AC, the diagonal of the paral-

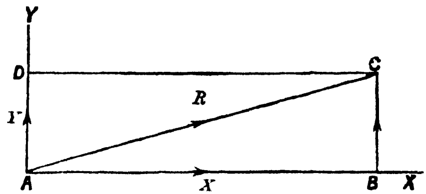


Fig. 33.

lelogram $ABCD$. Also, since X, Y are at right angles, the angle ABC is right.

Hence, by the definition of the tangent (Trig. 4),

$$\tan BAC = \frac{BC}{AB} = \frac{Y}{X} \quad \ldots\ldots\ldots\ldots\ldots \quad \mathbf{(5)}.$$

This determines the tangent of the angle BAC, and from it the angle itself may be found.

If $\tan BAC$ is $\frac{1}{3}\sqrt{3}$, 1, or $\sqrt{3}$, we know that the corresponding values of the angle BAC are 30°, 45°, and 60° respectively.

The *magnitude* of the resultant is R where

$$R^2 = X^2 + Y^2 \quad (\S\ 25).$$

Hence the resultant is completely determined both in magnitude and direction.

Example.—Two forces of $\sqrt{3}$ lbs. and 1 lb. include an angle of 150°. To find the direction and magnitude of their resultant.

Replace the forces by two forces X, Y acting along and perpendicular to the direction of the force of $\sqrt{3}$ lbs. Then, as in § 28,

$$X = \sqrt{3} - 1 \times \tfrac{1}{2}\sqrt{3} = \tfrac{1}{2}\sqrt{3} \text{ lbs.},$$

$$Y = \tfrac{1}{2} \text{ lb.};$$

$$\therefore \quad \frac{Y}{X} = \frac{1}{\sqrt{3}} = \tan 30^\circ;$$

$\therefore$ the resultant makes an angle 30° with the force of $\sqrt{3}$ lbs.

Also $\qquad R^2 = X^2 + Y^2 = \frac{1}{4} + \frac{3}{4} = 1$;

$\therefore$ resultant $R = 1$ lb.

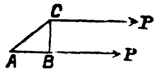


Fig. 34.

36. **Work.**—DEFINITION.—If a particle be moved from A to C under the action of a constant force P acting parallel to AB, and if CB be drawn perpendicular to AB, then the product

$$P \times (\text{distance } AB)$$

is called the **work done** by the force on the particle in changing its position. (*Dynamics*, § 199.)

The distance moved AC is called the *displacement* of the particle, and AB is called the *projection* of this displacement on the direction of the force.

OBSERVATION.—This definition holds good whether the displacement AC is large or small, provided that the force P remains constant while the particle is moving from A to C.

37. The work done by a force is the product of the displacement of its point of application into the resolved part of the force in the direction of that displacement.

Let AP represent any force P, and let the particle on which it acts be moved from A to C. Drop PM, CB per-

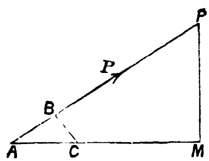


Fig. 35.

pendicular on AC, AP. Then AM represents the resolved part of P along AC. Since the angles ACP and AMP are right, a circle whose diameter is CP will pass through B, C, M, P, and therefore, by Euc. III. 36, Cor.,

$$AP \cdot AB = AC \cdot AM;$$

that is, $\quad P \times AB = AC \times (\text{resolved part of } P \text{ along } AC)$,

or, work done by force P = displacement × resolved part of P along direction of displacement.

38. **Principle of work for a single particle.**—*When a particle is moved from one position to another under the action of any number of forces, the algebraic sum of the works done by the several forces is equal to the work done by the resultant.*

For if the particle move from A to C, then algebraic sum of works done by the several forces

$= AC \times$ algebraic sum of resolved parts of the several forces along AC

$= AC \times$ resolved part of resultant along AC (§ 30)

$=$ work of resultant.

39. From this we get the following important corollary :—

COR. *If a particle, acted on by any number of forces in equilibrium, is moved from one position to another, the algebraic sum of the works done by the several forces is zero.*

For, in this case, the algebraic sum of the resolved parts of the forces along AC is zero, whence the result follows at once.

SUMMARY OF RESULTS.

Components of P along and perpendicular to line inclined at angle A are $X = P\cos A$, $Y = P\sin A$ ...(1), (§ 28.) and X is called the *resolved part* of P along the line. (§ 29.)

For $A = 30°$, $X = \frac{1}{2}\sqrt{3}\,.\,P$, $Y = \frac{1}{2}P$.

For $A = 60°$, $Y = \frac{1}{2}\sqrt{3}\,.\,P$, $X = \frac{1}{2}P$.

For $A = 45°$, $X = Y = \sqrt{\frac{1}{2}}\,.\,P$

If R is resultant of forces whose rectangular components are X_1, Y_1; X_2, Y_2, &c.,

$$R^2 = (X_1 + X_2 + \ldots)^2 + (Y_1 + Y_2 + \ldots)^2 \; \ldots \; (2). \quad (§\,30.)$$

For equilibrium both

$$X_1 + X_2 + \ldots = 0 \text{ and } Y_1 + Y_2 + \ldots = 0. \quad (§\,33.)$$

Magnitude of the resultant of any two forces P, Q is given by

$$R^2 = P^2 + Q^2 + 2PX \; \ldots\ldots \; (3), \quad (§\,31.)$$
$$= P^2 + Q^2 + 2PQ\cos(P, Q),$$

where $X =$ resolved part of Q along P,

and $(P, Q) =$ angle between forces P, Q.

If three forces P, Q, R balance.

$$\frac{P}{\sin(Q, R)} = \frac{Q}{\sin(R, P)} = \frac{R}{\sin(P, Q)} \ldots (4). \quad (\S\ 34.)$$

Direction of resultant of perpendicular forces X, Y makes with X an angle whose tangent $= Y/X$...... (5). (§ 35.)

Work done by P in displacement AC

$= P \times$ (projection of AC on P) (§ 36.)

$= AC \times$ resolved part of P along AC. (§ 37.)

Principle of Work.—Algebraic sum of works of components = work of resultant. (§ 38.)

EXAMPLES II.

1. Resolve the following forces into components, along and perpendicular to the straight lines to which they are inclined at the given angles:

(i.) 4 lbs., 30°; (iv.) 3 tons, 90°; (vii.) 8 cwt., 150°;
(ii.) $8\sqrt{2}$ oz., 45°; (v.) 12 grammes, 120°; (viii.) 4 mgr., 180°;
(iii.) 10 kilog., 60°; (vi.) 5 lbs., 135°; (ix.) 6 stone, 0°.

2. A force equal to 20 lbs. weight, acting vertically upwards is resolved into two forces, one of which is horizontal and equal to 10 lbs. weight. What is the magnitude and direction of the other component?

3. A force of $\sqrt{3}$ lbs. bisects the angle between two straight lines which include an angle of 60°. Find (i.) the components, (ii.) the resolved parts, of the force along these lines.

4. Find the magnitudes of the resultants of the following pairs of forces inclined at the given angles, namely—

(i.) 3 and 4 lbs., 0°; (vi.) 2 and 4 lbs., 60°,
(ii.) 10 and 24 grammes, 90°, (vii.) 5 and 10 lbs., 120°,
(iii.) 5 and 6 tons, 180°; (viii.) 4 and 12 mgr. 30°,
(iv.) 1 and $3\sqrt{2}$ kilog., 45°; (ix.) 4 and 6 oz., 150°.
(v.) $4\sqrt{2}$ and 1 cwt., 135°.

5. Two forces of 4 lbs. and 10 lbs. respectively act at a point and are inclined to each other at an angle of 60°. What is the magnitude of their resultant?

6. Indicate two forces, at right angles to each other, which could maintain equilibrium with the above (see Ex. 5).

7. Show how to find the resultant of two forces represented by the

diagonal of a square and one of the sides meeting the diagonal, both acting from the point where they meet. Prove that its magnitude $= \sqrt{5} \times$ a side.

8. A force of 10 lbs., acting northwards, is resolved into three components, of which one is 6 lbs. north-eastwards, and another 2 lbs. westwards. Find the third component.

9. A straight line COB has a line OA at right angles to it, and forces each of 7 lbs. act, one along OA, another along OB, and a third along the bisector of the angle COA. Find the magnitude of the resultant.

10. What is the resultant of three forces, 3, 4, and 5 lbs., acting at a point along lines making angles of 120° with each other?

11. If two forces P and Q act upon a particle (i.) when the angle between their directions is 60°, (ii.) when it is 120°, and if R and S are the resultants in these two cases, prove, geometrically if possible, that
$$R^2 + S^2 = 2(P^2 + Q^2).$$

12. R, R' are the smallest and greatest forces which, along with P and Q, can keep a particle at rest. Show that, if P, Q, $\sqrt{RR'}$ keep a particle at rest, two of the forces are perpendicular to each other.

13. Two forces, of magnitudes 1 and 3, have a certain resultant when their directions contain a certain angle; the square of the resultant is doubled if the direction of one of the forces is reversed. Find the resolved part of the former force along the direction of the latter.

14. Find the cosine of the angle between the directions of forces of 5 and 7 units, which have a resultant of 8 units. Show that the angle itself exceeds 90°.

15. A force of $10\sqrt{2}$ lbs. is inclined at angles 75° and 15° to two perpendicular straight lines. Find the resolved parts of the force along these lines by first replacing it by its components along the internal and external bisectors of the angles between them.

16. Calculate, to two places of decimals, the resultants of $5\sqrt{2}$ and 10 lbs. when the angle between them is 15° and when it is 105°; also the resultants of $2\sqrt{2}$ and 3 lbs. at an inclination of 75° and also of 165°.

17. If R be the resultant of forces $P+X$ and Y acting at right angles, write down the expression for R^2; and if X, Y are the resolved parts of a force Q along and perpendicular to the direction of P, deduce the formula $R^2 = P^2 + Q^2 + 2PX$.

EXAMINATION PAPER I.

1. What is meant by the *resultant* of two forces, and how can it be determined?

2. State the proposition known as the "Parallelogram of Forces," and describe an apparatus for verifying it experimentally.

3. Assuming the truth of the "Parallelogram of Forces," enunciate and prove the proposition known as the "Triangle of Forces."

4. Forces of 2, 3, and 4 lbs. act at a point O in directions parallel to the sides AB, AC, BC of an equilateral triangle, respectively. Find their resultant.

5. State the proposition known as the "Polygon of Forces." How far is the converse true?

6. If P be a point in a straight line AB such that $m.AP = n.PB$, and if O be any other point, prove that two forces represented by $m.OA$ and $n.OB$ have a resultant represented by $(m+n).OP$.

7. Show that a force may be resolved into two components in any number of different ways, and explain what is meant by the *resolved part* of a force in any given direction.

8. $ABDC$ is a parallelogram; E is a point in CD. Find, when possible, a point F in AB such that the magnitude of the resultant of two forces represented by AE, AF may be represented in magnitude by AD.

9. State and prove the formula giving the magnitude of the resultant of any two forces in terms of the components and the resolved part of one component along the line of action of the other.

10. Three forces act at a point and keep equilibrium; find their ratios, having given the angles between them.

CHAPTER III.

THE INCLINED PLANE.

PARTICULAR CASES OF EQUILIBRIUM.

40. **Equilibrium on a smooth inclined plane.**—The conditions of equilibrium of a weight resting on an inclined plane may be found very readily by means of either the Triangle of Forces or the Principle of Work,

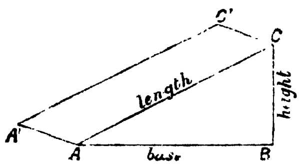


Fig. 36.

when the weight is either pushed against the plane by a horizontal force, or is supported by a force acting along the plane.

The force employed to support or raise the weight is sometimes called the *effort* or *power* (see § 83).

In diagrams it is usual to represent an inclined plane by its section *ABC*, and *not* in perspective as in Fig. 36.

41. Equilibrium on an inclined plane under a supporting force applied *horizontally*.

Let a body of weight W be supported at any point O on the plane ABC by a horizontal force P.

It is required to find P, the dimensions of the section ABC being supposed given.

The three forces which keep the body in equilibrium are:

(i.) The weight W acting vertically downwards, and therefore perpendicular to AB.

(ii.) The applied force P acting horizontally, and therefore perpendicular to BC.

(iii.) The reaction of the plane, acting perpendicular to CA. Let this reaction be denoted by R.

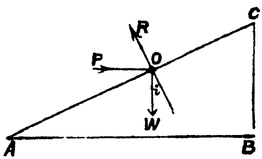


Fig. 37.

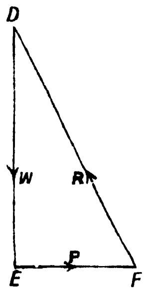


Fig. 38.

Therefore the three forces W, P, R act perpendicular to the sides AB, BC, CA taken in order.

Turn the inclined plane round, through a right angle, into the position DEF, so that its base DE is now vertical and its height EF horizontal (Fig. 38). Then the forces W, P, R are parallel to DE, EF, FD.

Let the length DE be taken to represent the weight W in magnitude as well as in direction.

Then, by the Triangle of Forces, the three sides of the triangle DEF represent the three acting forces W, P, R both in direction and in magnitude.

Therefore $$\frac{P}{EF} = \frac{W}{DE} = \frac{R}{FD}.$$

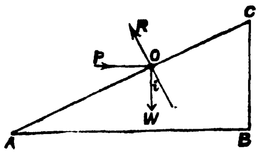


Fig. 37.

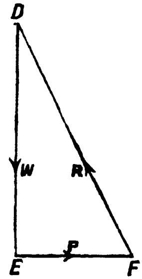


Fig. 38.

Also the triangles DEF, ABC are equal in all respects;

therefore $$\frac{P}{BC} = \frac{W}{AB} = \frac{R}{CA}.$$

Therefore $$P = W \times \frac{EF}{DE} = W \times \frac{BC}{AB};$$

or $$P = W \times \frac{\textbf{height of plane}}{\textbf{base of plane}} \quad \cdots\cdots\cdots \quad (\mathbf{1}).$$

Also $$R = W \times \frac{FD}{DE} = W \times \frac{CA}{AB};$$

or $$R = W \times \frac{\textbf{length of plane}}{\textbf{base of plane}} \quad \cdots\cdots\cdots \quad (\mathbf{2}).$$

Since action and reaction are equal and opposite (by Newton's Third Law), the weight presses against the plane with a force equal and opposite to the reaction of the plane, and the magnitude of this force of pressure is therefore R and is given by (2).

42. **Alternative method.** — *The expression for the supporting force P also follows very simply from the Principle of Work.*

Let the force P, acting horizontally, push the weight up the plane from A to C with uniform velocity.* Then the horizontal and vertical distances traversed by the weight are AB, BC respectively; hence the works done by P and against W are $P \times AB$ and $W \times BC$. Therefore, equating these, we have, by the Principle of Work,

$$P \times AB = W \times BC;$$

$$\therefore \quad P = W \times \frac{BC}{AB} = W \times \frac{\text{height}}{\text{base}} \quad \text{.................} \quad (\mathbf{1}),$$

as before.

[It would be less easy to determine the reaction R by means of the Principle of Work.]

43. *Trigonometrical Expression.*—If A denote the angle of inclination BAC, we have

$$\tan A = \frac{BC}{AB};$$

$$\therefore \quad P = W \tan A \quad \text{..........................} \quad (1a).$$

[This relation may also be found by equating the resolved parts of P and W along the plane. For P makes an angle A with the plane, while W makes an angle A with a line *perpendicular* to the plane. Hence we obtain

$$P \cos A = W \sin A;$$

$$\therefore \quad P = W \frac{\sin A}{\cos A} = W \tan A. \quad \text{[Trig. § 14.]}$$

In like manner, by resolving vertically, we should have

$$R \cos A = W,$$

or

$$R \times \frac{AB}{AC} = W,$$

a result in accordance with (2)].

Cor. *The following results should be verified by the student as an exercise:—*

If the inclination of the plane is

$$0°, \quad 30°, \quad 45°, \quad 60°,$$

the horizontal force required to support W is

$$0, \quad \sqrt{\tfrac{1}{3}}W, \quad W, \quad \sqrt{3}W,$$

and the force of pressure on the plane is

$$W, \quad \sqrt{\tfrac{4}{3}}W, \quad \sqrt{2}W, \quad 2W.$$

---

* If the velocity were not uniform, the forces W, P, R would not be in equilibrium, and, moreover, we should have to take account of the work done in producing kinetic energy

Example.—To find the horizontal force which will support a weight of half a ton on an incline of 30°.

Here $$\angle BAC = 30°,$$

$$\therefore \quad BC = AB \tan 30° = AB \times \tfrac{1}{3}\sqrt{3};$$

also $W = \frac{1}{2}$ ton $= 10$ cwt., and the applied force P acts horizontally;

$$\therefore \quad P = W \times \tfrac{1}{3}\sqrt{3} = \tfrac{10}{3}\sqrt{3} \text{ cwt.},$$

or required force $= 5{\cdot}7735 \ldots$ cwt.

$= 5$ cwt. 3 qrs. $2\frac{3}{4}$ lbs. wt. nearly.

44. Equilibrium on an inclined plane, the supporting force acting *along the plane*.

Let a given weight W rest on a smooth inclined plane of given section ABC, and let it be kept from sliding down by a force P acting up the plane.

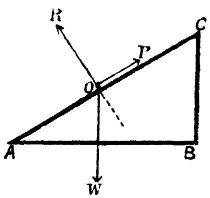


Fig. 39.

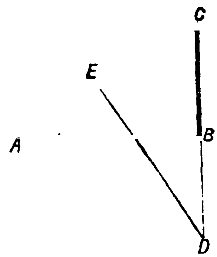


Fig. 40.

It is required to find the magnitude of P.

Let R be the reaction of the plane.

Then the forces acting on the weight are

P, acting in the direction AC;

W, acting vertically downwards;

R, acting perpendicular to the plane (since the plane is smooth).

Produce the vertical CB to D, and make $CD = CA$. Also draw DE perpendicular on the plane. Then the triangles ABC, DEC are equal in all respects, and therefore

$$BC = EC, \quad AB = DE.$$

Now, the forces P, W, R are parallel to the sides EC, D, DE of the triangle DEC. Therefore, by the Triangle of Forces,

$$\frac{P}{EC} = \frac{W}{CD} = \frac{R}{DE};$$

whence

$$\frac{P}{BC} = \frac{W}{CA} = \frac{R}{AB}.$$

Therefore

$$P = W \times \frac{BC}{CA}$$

$$= W \times \frac{\textbf{height of plane}}{\textbf{length of plane}} \quad \ldots\ldots\ldots \quad \mathbf{(3)},$$

and

$$R = W \times \frac{AB}{CA}$$

$$= W \times \frac{\textbf{base of plane}}{\textbf{length of plane}} \quad \ldots\ldots\ldots \quad \mathbf{(4)}.$$

45. **Alternative method.**—*The expression for P also follows very simply from the Principle of Work.*

Let the force P applied along an inclined plane pull the weight W from the bottom to the top of the plane with uniform velocity. Then P moves its point of application along the length AC, and the weight W is raised against gravity through the vertical height BC of the plane. Equating the two amounts of work, we have

$$P \times \text{length of plane} = W \times \text{height of plane},$$

$$P = W \times \frac{\text{height of plane}}{\text{length of plane}} \quad \ldots\ldots \ \ldots\ldots \ \ldots\ldots \quad (3).$$

Example.—A road rises 440 feet in a mile. To find the pull that a horse must exert on a cart weighing 6 cwt. to draw it up the road.

Let the force be P cwt. Then work done by P in moving its point of application through 1 mile = work required to lift 6 cwt. through 440 feet;

$$\therefore \quad P \times 5280 = 440 \times 6;$$

$$\therefore \quad P = \frac{6 \times 440}{5280} = \frac{6}{12} = \tfrac{1}{2} \text{ cwt.} = 56 \text{ lbs. wt}$$

46. *Trigonometrical Expressions.*—Taking $\angle BAC = A$,

we have

$$\sin A = \frac{BC}{AC}, \quad \cos A = \frac{AB}{AC};$$

therefore
$$P = W \sin A \text{ .... } (3a),$$
$$R = W \cos A \text{ .... } (4a).$$

[These expressions may also be found by resolving along the plane and perpendicular to it, for the resolved parts of W perpendicular and along the plane are $W \cos A$ and $W \sin A$, and these are equal respectively to P and R.]

COR. *The following results should be verified by the student as an exercise :—*

If the inclination of the plane is

$$0, \quad 30°, \quad 45°, \quad 60°, \quad 90°,$$

the force up the plane which will support W is

$$0, \quad \sqrt{\tfrac{1}{4}}\,W, \quad \sqrt{\tfrac{2}{4}}\,W, \quad \sqrt{\tfrac{3}{4}}\,W, \quad \sqrt{\tfrac{4}{4}}\,W,$$

and the force of pressure on the plane is

$$\sqrt{\tfrac{4}{4}}\,W, \quad \sqrt{\tfrac{3}{4}}\,W, \quad \sqrt{\tfrac{2}{4}}\,W, \quad \sqrt{\tfrac{1}{4}}\,W, \quad 0.$$

Example.—To find the force acting up an incline of 30° that will support a weight of $\frac{1}{2}$ cwt.

In this case the required force

$$= \text{weight} \times \frac{\text{height of plane}}{\text{length of plane}} = \tfrac{1}{2} \text{ cwt.} \times \sin 30°$$
$$= \tfrac{1}{4} \text{ cwt.} = 28 \text{ lbs. wt.}$$

47. **Equilibrium on an inclined plane, the supporting force being applied in any direction whatever.**

When the supporting force P is applied in any direction other than those considered above, its magnitude can, in general, only be calculated by Trigonometry, but it may be determined graphically thus.

On the vertical through O measure OD downwards containing as many units of length as there are units of force in the weight W. Draw DF perpendicular to the inclined plane, and let it meet the line in which P is applied in the point E. Then, by the Triangle of Forces, EO represents the force P, and DE represents

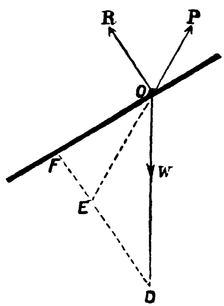


Fig. 41.

the reaction R. Hence, if the figure is carefully drawn, P and R can be found by measuring the lengths EO, DE.

For different directions of P, the point E always lies on the straight line DF. Evidently EO is least when E is at F, because the perpendicular OF is less than any other straight line drawn from O to the line DE.

Hence *the force required to support a given weight is least when it acts along the plane.*

48. **The Triangle of Forces** can often be applied to the equilibrium of weights supported by strings, rods, or inclined planes, when it is required to calculate the supporting forces. In drawing a diagram to represent these, it frequently happens that certain lines naturally form a triangle of forces, and the problem is then very simple.

The following may be taken as types of such problems:-

Examples.—(1) In the crane ACB, the jib or rod CA is 12 ft. long, and is connected to the wall BC by a chain AB, 8 ft. long, attached at a point B 6 ft. above C.

To find T the pull of the chain and P the thrust of the rod, when a weight W, equal to 18 cwt., is hung from A.

The forces at A are

(i.) T along AB,

(ii.) P along CA,

(iii.) W or 18 cwt. acting vertically, that is, is parallel to BC.

Hence these forces are parallel to the sides of the triangle ABC.

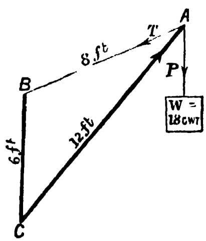


Fig. 42.

Therefore, by the Triangle of Forces, T, P, W can be represented in magnitude by AB, CA, BC.

But $\qquad AB = 8$ ft., $\quad CA = 12$ ft., $\quad BC = 6$ ft.;

$$\therefore \quad AB = \tfrac{4}{3}BC \quad \text{and} \quad CA = 2BC;$$

$$\therefore \quad T = \tfrac{4}{3}W \quad \text{and} \quad P = 2W.$$

But $\qquad W = 18$ cwt.;

$$\therefore \quad \text{tension of chain } T = \tfrac{4}{3} \times 18 = 24 \text{ cwt.,}$$

and $\qquad$ thrust of jib $P = 2 \times 18 = 36$ cwt.

(2) **A string is attached to two pegs A, C in a horizontal line 24 ft. apart, and a weight of 10 lbs. is suspended from its middle point B. If this point falls 5 ft. below the line AC, to find the tension in the string.**

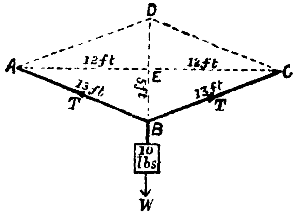


Fig. 43.

Complete the parallelogram $ABCD$.

Since $AB = BC$, BD evidently bisects AC at right angles in E, and BED is vertical.

Let BA, BC represent the tensions in the two portions of the string. Then, by the Parallelogram of Forces, DB represents the weight of 10 lbs.

Now, we are given that $BE = 5$ ft., $EC = 12$ ft.

$$\therefore \quad BC^2 = BE^2 + EC^2 = 5^2 + 12^2 = 13^2; \quad \text{or} \quad BC = 13 \text{ ft.}$$

Also $$BD = 2BE = 10 \text{ ft.}$$

Since BE or 10 ft. represents a force of 10 lbs.,

$\therefore$ BC, which is 13 ft., represents a force of 13 lbs.

Therefore the tension of the string = 13 lbs.

(3) **AB and AC are two chains 9 ft. and 12 ft. long attached to pegs B, C at a horizontal distance of 15 ft. apart. To find the pulls in the chains when a weight of 1 ton is suspended from A.**

Here the lengths AB, AC, BC are proportional to 3, 4, 5;

$$\therefore \quad BC^2 = BA^2 + AC^2:$$

therefore BAC is a right angle (by Euc. I. 48).

Let P be the pull in AB, Q that in AC, and let $W = 1$ ton, the weight at A. Then P acts perpendicular to CA, Q perpendicular to AB, and W acts perpendicular to BC. Hence the three forces at A act perpendicular to the sides of the triangle ABC, and if this triangle is turned through a right angle, its sides, taken in order, will be brought parallel to the forces, as in Fig. 45. Therefore, by the Triangle of Forces, P, Q, W are represented in magnitude by CA, AB, BC, that is, by 9, 12, 15 ft. respectively.

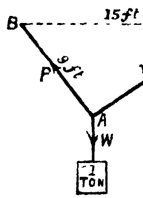


Fig. 44.

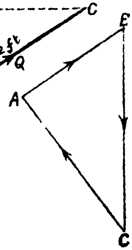


Fig. 45.

Therefore $$P = \tfrac{12}{15} W = \tfrac{4}{5} \text{ of a ton} = 16 \text{ cwt.},$$
$$Q = \tfrac{9}{15} W = \tfrac{3}{5} \text{ of a ton} = 12 \text{ cwt.}$$

(4) A weight of 1 lb. is suspended by a string. To find the angle through which the string will be pulled aside out of the vertical by horizontal force of $\sqrt{3}$ lbs., and to find the pull in the string.

Let T be the required pull in the string, W the weight ($=$ 1 lb.), F the horizontal force ($= \sqrt{3}$ lbs.).

Let ***PK*** be the position of the string, the weight being at ***K***. Then if the horizontal line ***KM*** meets the vertical ***PM*** at ***M***, the forces W, F, T are represented in direction by ***PM***, ***MK***, ***KP***, and therefore they can also be represented in magnitude by these lines.

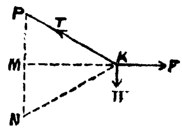


Fig. 46.

Now $$F = \sqrt{3}W;$$

$$\therefore \quad MK = \sqrt{3}PM;$$

$$\therefore \quad \angle MPK = 60°,$$

and $$\angle MKP = 30°.$$

Therefore the string makes an angle 60° with the vertical.

Also $$PK = 2PM;$$

$\therefore$ the required pull $T = 2W = 2$ lbs. wt.

(5) A weight of 1 ton is attached at ***B*** to a rod ***AB***, which is drawn aside from the vertical position through 30° by a chain ***BD*** attached to ***B***. Find the pull in the rod, supposing ***BD*** to make an angle of 60°, (i.) with the upward drawn vertical, (ii.) with the downward drawn vertical.

Let R be the required pull in the rod ***BA***, P the pull in the chain ***BD***, Q ($=$ 1 ton) the given weight

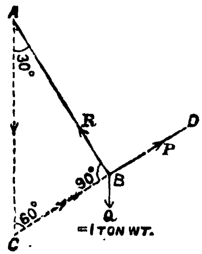


Fig. 47.

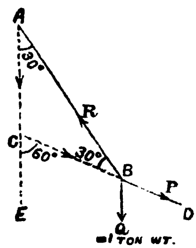


Fig. 48.

Take any point ***A*** on the rod, and let the vertical through ***A*** meet ***DB*** produced in ***C***.

Then the forces P, Q, R are represented in direction by ***CB***. ***AC***, ***BA***, Therefore they can also be represented in magnitude by these lines.

(i.) In the first figure,

$$\angle BAC = 30°, \quad \angle ACB = 60°; \text{ and } \therefore \quad \angle CBA = 90°.$$

$$\therefore \quad BA = \frac{\sqrt{3}}{2} AC.$$

$$\therefore \text{ required pull } R = \frac{\sqrt{3}}{2} Q = \frac{\sqrt{3}}{2} \text{ tons weight.}$$

(ii.) In the second figure,

$$\angle BAC = 60°, \quad \angle ECB = 30°; \text{ and } \therefore \quad \angle CBA = 30°.$$

Therefore ACB is an isosceles triangle having its base angles each 30°, and if C be joined to the middle point of AB, the triangle ACB will be divided into two triangles whose angles are 30°, 60°, 90°. (Trig., § 19.)

$$\therefore \quad AB = 2 \times \frac{\sqrt{3}}{2} AC = \sqrt{3}\, AC.$$

$$\therefore \text{ required pull } R = \sqrt{3}\, Q = \sqrt{3} \text{ tons weight.}$$

Summary of Results.

For equilibrium of weight W on a *smooth inclined plane*, if supporting force P be *horizontal*,

$$P = W \times (\text{height}) \div (\text{base}) \ldots (1), \quad (\S\ 41.)$$

$$\text{reaction } R = W \times (\text{length}) \div (\text{base}) \ldots (2). \quad (\S\ 41.)$$

Work done in drawing W up plane

$$= W \times (\text{height}) = P \times (\text{base}).$$

[In terms of the inclination A,

$$P = W \tan A \ \ldots\ldots (1a). \quad (\S\ 43.)]$$

If P acts *up the plane*, *i.e.*, in the most favourable direction (§ 47), then

$$P = W \times (\text{height}) \div (\text{length}) \ldots (3), (\S\ 44.)$$

$$R = W \times (\text{base}) \div (\text{length}) \ldots (4). \ (\S\ 44.)$$

Work done in drawing W up plane

$$= W \times \text{height} = P \times \text{length}.$$

[In terms of the inclination A,

$$P = W \sin A, \quad R = W \cos A. \quad (\S\ 46.)]$$

EXAMPLES III.

1. Find (*a*) the horizontal forces, and (*b*) the forces up the plane required to support each of the following weights on the given inclined planes:—

(i.) 10 lbs. on an incline of length 10 ft. and height 6 ft.;
(ii.) 78 lbs. on an incline of height 5 ft. and base 12 ft.;
(iii.) 30 tons on an incline of length 25 yds. and base 24 yds.;
(iv.) 85 kilogs. on an incline of 8 in 17 of length 34 metres.

2. Find also the works done in drawing the weights of the last question up the planes.

3. Find the horizontal force, and find also the force acting up the plane, required to support a weight of

(i.) 5 tons on an incline of 30°;
(ii.) 28 lbs. on an incline of 45°;
(iii.) 10 kilogs. on an incline of 60°.

In each case find the reaction of the plane.

4. The lengths of the three inclined planes of the preceding question are, respectively, (i.) 9 ins., (ii.) 3 ft., (iii.) 150 centimetres. Find the works done in drawing the weights up the planes.

5. A weight of 9 lbs. is pulled up an inclined plane of which the height is $1\frac{1}{2}$ ft. and the base is 2 ft. What force (i.) acting horizontally, (ii.) acting along the plane, is required for the purpose? What amount of work is done in each case?

6. A weight of 12 lbs. is supported by two strings, each inclined at an angle of 45° to the vertical. Find the tension of the strings.

7. A weight of 10 lbs. is supported by two strings which are inclined at angles of 30° to the vertical. Find the tensions in the strings.

8. A stone weighing 1 ton is suspended in the air by a chain; a rope fastened to the stone is pulled so that the chain makes 30° and the rope 60° with the vertical. Draw a very careful figure showing the three forces acting on the stone, and a triangle representing them. Find the pull on the rope.

9. A small heavy ring, which can slide freely upon a smooth thin rod, is attached to the end of the rod by a fine string. If the rod be held in any position inclined to the vertical, draw a triangle representing the forces acting upon the ring.

10. A body of weight 10 lbs. rests on a smooth plane inclined at an angle of 30° to the horizontal. Find the least value of the force required to sustain it and the reaction of the plane.

11. A weight rests on a smooth inclined plane. Determine the direction and magnitude of the least force which will keep it in equilibrium Find also the direction of the force in order that the thrust on the plane may be double of that exerted in the first case.

12. Two weights support each other on two smooth inclined planes of the same height, the weights being connected by a string passing over a smooth pulley at the junction of the planes. The angles of the planes are 30° and 60°. What is the ratio of the weights and the tension of the string?

13. A wedge, whose triangular section ABC is right-angled at A, is placed with BC on a horizontal plane; a horizontal force P will support a weight Q on AB. What horizontal force would be required to support a weight Q placed on AC?

14. A picture, weighing 56 lbs., is slung over a nail in the ordinary way by a cord attached to two eyes in the top horizontal bar of its frame. If the height of the nail above this bar is half the distance between the eyes, what is the tension in the cord? Under what circumstances would the tension be equal to or greater than the whole weight of the picture?

15. A particle P is placed inside a smooth circular tube, and is acted on by two forces towards the extremities A and B of a fixed diameter AB. The forces are respectively proportional to PA and PB. Find the position of equilibrium.

16. A wedge, whose triangular section ABC is right-angled at A, is placed with BC on a horizontal plane. Two weights P and Q, connected by a string, will balance if placed with the string passing over A and P resting on AB, and Q on AC. If AB be placed horizontally, they will balance with P on BC and Q hanging vertically. Show that the sides of the triangle are in a geometrical progression whose common ratio is equal to the ratio $P : Q$.

17. Determine the tension of the threads of a rectangular piece of network hung from a horizontal bar, due to suspending a series of equal weights in a horizontal line at the lowest points of the net, supposing the meshes are equal hexagons of which a pair of sides are vertical.

CHAPTER IV.

THE TRANSMISSION OF FORCE—EQUILIBRIUM OF THREE FORCES.

49. **Rigid bodies.** — In treating the conditions of equilibrium of several forces acting "*at a point*," we have supposed the forces to be all applied to a single particle placed at that point. When two or more forces act *in parallel straight lines*, it is impossible to suppose them to be applied to the same particle, for parallel lines never meet. They must, therefore, be supposed to be applied to a body of extended size. Accordingly, it will be necessary to state what is meant by a rigid body before proceeding further.

DEFINITION.—*A* **rigid body** *is a body whose size and shape always remain the same whatever forces be applied to different parts of it.*

By this it is implied that *the distance between any two particles of a rigid body always remains the same.*

In reality no body is perfectly rigid, but most solid bodies may be regarded as rigid for all practical purposes.

50. **Rotation.**—A rigid body can be **rotated.**

When a top is spinning, the different particles of the top *rotate* or move round and round rapidly, though the top does not change its position as a whole. The same is true when a wheel *rotates* or turns round. When a door is opened, the hinge remains where it is and the other parts of the door *rotate* or turn about it, those furthest from the hinge moving most rapidly.

When a body changes its position as a whole, this motion is said to be a motion of *translation* to distinguish it from rotation. All the motions treated of in Book I. (Dynamics) are motions of translation.*

* That branch of Mechanics which deals with rigid bodies rotating under the action of forces is called Rigid Dynamics, and cannot be treated satisfactorily by elementary methods.

When a number of forces acting on a rigid body are in equilibrium, they must have no tendency to produce motion *either of translation or rotation*; otherwise the body will not remain at rest.

51. **Transmission of Force.** — In the first place, it will be observed that

Two forces acting at two points of a rigid body are in equilibrium if, and only if, they are equal and opposite, and act in the same straight line.

This may readily be verified by attaching strings to two points A, B of a body (say a stick) resting on a horizontal table, and then

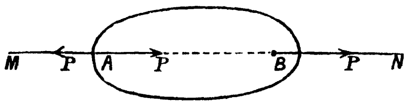


Fig. 49.

pulling the strings horizontally. The body will turn round until the strings MA, BN are both in one straight line, and will then come to rest. And, since the forces produce no motion of the body as a whole (*i.e.*, no motion of translation), they must be equal and opposite.

[If the body is again displaced so as to bring the strings out of one straight line, it will not remain at rest, but will rotate back to its former position. Hence two equal and opposite forces which do not act in the same straight line are not in equilibrium, but tend to produce rotation.]

52. From the above property we deduce the following principle, which is known as

The principle of Transmissibility (or Transmission) of Force:

A force acting on a rigid body may have its point of application transferred to any point whatever in the straight line in which it acts without affecting the conditions of equilibrium.

Let P be any force applied to a body at B in the direction BN. Let A be any point of the body in the straight line BN or BN produced. At A apply two equal

and opposite forces of magnitude P in the straight line AB. These two forces balance each other, and therefore do not affect the conditions of equilibrium of the original forces. Now consider the two forces $+P$ at B and $-P$ at A. By the property just proved, these two forces balance each other, and therefore they can be removed without affecting the conditions of equilibrium. We are, therefore, left with the force $+P$ at A as the statical equivalent of the original force P at B applied in the same straight line; as was to be proved.

The principle of the Transmission of Force may also be stated thus.

When a force acts on a rigid body, it is immaterial what point in its line of action is considered to be the point of application of the force.

The point of application may even be taken *outside the body*, provided that the force is applied to a particle rigidly connected with the body. But a force *cannot* be replaced by an equal and parallel force acting at any point *not* in its original line of action.

53. Conditions of equilibrium of three forces in one plane.

If a rigid body is in equilibrium under three forces in one plane, their lines of action must all be parallel or all pass through one common point.

For let the three forces be not all parallel. Then the lines of action of two of them must meet in some point, say O. By the principle of Transmission of Force, we may suppose these two forces to be applied to a particle of the body (or rigidly connected with the body) at O. Hence, as in Chap. I., they are equivalent to a single resultant force acting at O. This resultant and the third force balance; therefore they must be equal and opposite and in the same straight line (§ 51). Hence the

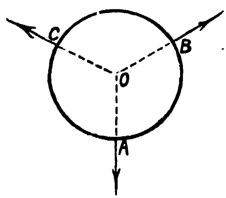


Fig. 50.

line of action of the third force must pass through O, and therefore the three lines of action must all pass through one common point.

In addition to passing through one common point, the three forces must satisfy the same conditions of equilibrium as if they acted on a particle at that point; they must therefore be capable of being represented in magnitude and direction by the sides of a triangle taken in order.

The conditions of equilibrium of three *parallel* forces will be investigated in Chap. VI.

54. *The point of intersection of the forces need not be within the body.*

Thus, let three cords be attached to a ring or hoop at the equidistant points A, B, C (Fig. 50), and let these cords be pulled with equal forces in the direction of radii OA, OB, OC. Then these forces will be in equilibrium, for their directions pass through one common point (viz., the centre O), and are inclined at angles of 120°. Hence the ring will remain at rest notwithstanding the fact that the point O is not in the substance of the ring itself. (See also § 57, Ex. 1.)

55. **The wedge** is a triangular block which is used either for splitting a body (*e.g.*, a piece of wood) into two parts, for separating two bodies, or for slightly raising heavy weights off the ground. The section of the block is a triangle, and the wedge generally studied on account of its greater utility and simplicity is isosceles, and could be formed by two right-angled inclined planes put back to back. A knife and a chisel afford good illustrations of the principle of the wedge.

The wedge we consider in theoretical mechanics is smooth; but most wedges are so rough that, when they have been driven in between two bodies, the friction prevents them from coming out again. Usually, too, wedges are driven in by blows of a hammer; hence the conditions of equilibrium found below are usually far from being realized in practice.

56. **Conditions of equilibrium of a smooth wedge.** —Let a smooth wedge, whose section is the triangle ABC, be driven in between two bodies M, N by a force P

applied on its face BC. Let Q, R be the reactions with which M, N resist the insertion of the wedge.

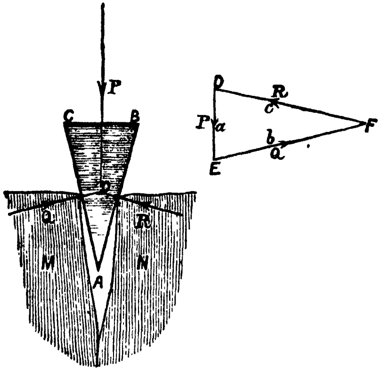


Fig. 51.

Then the wedge is kept in equilibrium by the three forces P, Q, R, and, since the wedge is smooth, these forces act perpendicular to BC, CA, AB. Hence, if the triangle ABC be turned, through a right angle, into the position DEF, its sides will be brought parallel to the forces.

Therefore P, Q, R can be represented in magnitude by BC, CA, AB, respectively, that is,

$$\frac{P}{BC} = \frac{Q}{CA} = \frac{R}{AB};$$

or, if a, b, c denote the lengths of BC, CA, AB,

$$\frac{P}{a} = \frac{Q}{b} = \frac{R}{c} \quad \text{..................} \quad (1).$$

In the case generally considered where the section of a wedge is isosceles, $b = c$. We have

$$Q = R = P \times \frac{b}{a}.$$

By sufficiently increasing the length b and making the breadth a very small, a very small force P can be made to overcome a very large resistance Q. A hatchet is a good example of this.

Example.—A stone rests against a vertical wall AC, and can be separated from the wall by a smooth wedge weighing 28 lbs., whose triangular section ACB is right-angled at C, and has its sides 7, 24, 25 inches long. To find the horizontal force with which the stone presses against the wedge, no other forces being applied to the wedge.

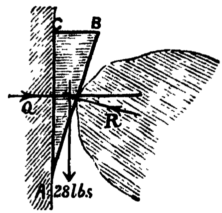


Fig. 52.

Let Q be the reaction of the wall, R the force of pressure of the stone on the wedge, P (= 28 lbs.) the weight of the wedge. Then the wedge is kept in equilibrium by forces P, Q, R.

Since ACB is a right angle, CB is horizontal, and therefore P acts perpendicular to BC. Since the wedge is smooth, Q, R act perpendicular to CA, AB respectively. Therefore, by the Triangle of Forces,

$$\frac{P}{BC} = \frac{Q}{CA} = \frac{R}{AB},$$

or

$$\frac{28}{7} = \frac{Q}{24} = \frac{R}{25};$$

whence $Q = 96$ lbs., $R = 100$ lbs.

Also, by resolving horizontally, we have

required horizontal component of $R = Q = 96$ lbs.;

therefore the stone presses with a horizontal force of 96 lbs.

57. **Equilibrium of a heavy body.—Applications to problems.**—The theorem of § 53 is very useful in enabling us to find the conditions of equilibrium of a heavy body supported at two given points by forces that are not vertical. The cases where the supporting forces are vertical will be considered later.

It will be proved in Chap. XI. that the whole weight of a rigid body may always be supposed to act vertically at a single point of the body, called its *centre of gravity* or *centre of mass*. For the present, the following particular results will be assumed:—

(1) The weight of a heavy *uniform rod* or *beam* acts at its middle point.

(2) The weight of a *uniform sphere* or *cube*, or of a *circular disc*, acts at its centre.

It will also be necessary to remember that—

Action and reaction are equal and opposite (Newton's Third Law); and that

The reaction of a perfectly smooth surface is always perpendicular to that surface.

58. *Examples.*—(1) **Equilibrium of a ladder.**—A uniform ladder of weight W leans against a perfectly smooth wall. To find the thrusts which it exerts against the wall and ground when the ladder is 20 ft. long and reaches a height of 16 ft.

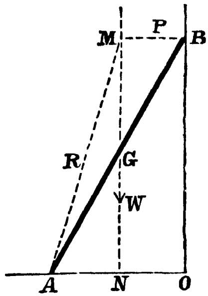


Fig. 53.

Let AB be the ladder, G its middle point.

Let P denote the reaction of the wall, R that of the ground.

Then the three forces acting on the ladder are—

(i.) Its weight W acting vertically through G (since the ladder is uniform);

(ii.) The reaction P at B acting horizontally (since the wall is smooth and vertical);

(iii.) The reaction R acting at A.

Since these forces are in equilibrium, they must pass through one point. Let the vertical through G meet the horizontal through B in M. Then the reaction R must act in the line AM.

Let MG meet the ground in N. Then the forces W, P, R act in the directions of MN, NA, AM, the sides of the triangle MAN taken in order. Therefore, by the Triangle of Forces, these sides may be made to represent the forces in magnitude, so that if BC or MN represents W, NA and AM will represent P and R respectively. Therefore

$$P = W \times \frac{NA}{MN}, \quad R = W \times \frac{AM}{MN}.$$

Now $$AB = 20 \text{ ft.}, \quad BC = 16 \text{ ft.};$$

$$\therefore \quad AO^2 = 20^2 - 16^2 = 4^2(5^2 - 4^2) = 4^2 \cdot 3^2 = 12^2;$$

$$\therefore \quad AO = 12 \text{ ft.};$$

and it is easily seen* that

$$AN = \tfrac{1}{2}AO = 6 \text{ ft.}$$

Therefore also

$$AM^2 = AN^2 + NM^2 = 6^2 + 16^2 = 2^2(3^2 + 8^2) = 2^2 \times 73;$$

$$\therefore \quad AM = 2\sqrt{73} \text{ ft.}$$

$$\therefore \quad \text{Reaction of wall } P = W \times \frac{6}{16} = \frac{3W}{8},$$

$$\text{Reaction of ground } R = W \times \frac{2\sqrt{73}}{16} = \frac{W\sqrt{73}}{8}.$$

* For $AG = GB$; $\therefore$ triangles AGN, BGM are equal in all respects; $\therefore$ $AN = BM = NO$ (the opposite side of the parallelogram $MBON$); $\therefore$ $AN = \tfrac{1}{2}AO$.

(2) **A uniform rod AB weighing 1 cwt., hinged at A, is supported in a horizontal position by a rope attached to B, and making an angle of 45° with the rod. To find the tension in the rope and the force at the hinge.**

Let G be the middle point of AB.

Let P be the tension in the rope, Q the force at the hinge, and let W denote the weight of the beam (1 cwt.).

Then the forces acting on the rod are—

(i.) Its weight 1 cwt. acting vertically through G;

(ii.) The tension P acting along the string at B;

(iii.) The reaction R of the hinge at A.

These three forces must all pass through one point. Let this point be C. Draw AD parallel to CB.

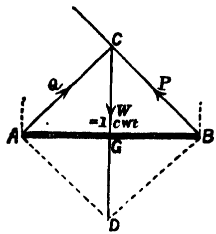


Fig. 54.

Then P, Q, W are represented in direction by DA, AC, CD.

Therefore they may also be represented in magnitude by these lines.

Since $\angle ABC = 45°$ and $\angle CGB = 90°$, therefore $\triangle GBC$ is a right-angled isosceles triangle. Since $GA = GB$, the triangles GAC, GBC, GAD are easily seen to be equal in all respects. Hence every triangle in the figure is a right-angled isosceles triangle, and every angle in the figure is either 90° or 45°. In the triangle DAC,

$$DA = AC = \tfrac{1}{2}\sqrt{2} \, . \, AD;$$

$$\therefore \quad P = \quad Q = \tfrac{1}{2}\sqrt{2}\,W = \sqrt{2} \text{ cwt.}$$

Therefore the tension in the string is $\frac{1}{2}\sqrt{2}$ of a cwt., and the force at the hinge is also $\frac{1}{2}\sqrt{2}$ of a cwt., and its direction makes an angle 45° with the horizon.

[As an exercise the student should work out the case where the string is in the straight line DB produced, making an angle of 45° in the opposite direction with AB. It will readily be found that the tension of the string and the force at the hinge are each $\frac{1}{2}\sqrt{2}$ cwt., as before, and that the latter force acts in the direction DA.]

(3) A ball 1 ft. in radius, and weighing 5 lbs., rests against a smooth wall, and is attached to a string which passes through a hole in the wall at A, and is pulled with a force of 13 lbs. To find the length of string projecting from the hole.

Let R be the reaction of the wall.

The forces on the ball are—

(i.) Its weight 5 lbs. acting vertically through its centre C;

(ii.) The reaction R acting perpendicular to the wall, and therefore in direction BC;

(iii.) The pull of the string or 13 lbs.

Since these pass through a point, the direction of the string passes through C, and the three forces are represented in direction by AB, BC, CA.

Therefore, if the length AB represents the weight 5 lbs., BC will represent the reaction R and CA the pull of 13 lbs.

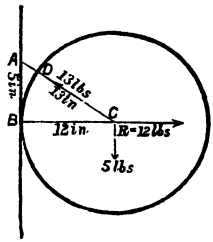


Fig. 55.

Now $CAB = 90°$;

$\therefore\ AB^2 + BC^2 = AC^2$,

or $5^2 + R^2 = 13^2$;

whence $R = 12$ lbs.

But $BC = 12$ lbs.;

$\therefore$ a force of 1 lb. is represented by 1 inch;

$\therefore\ AB = 5$ inches and $CA = 13$ inches;

and, if D is the point of attachment of the string,

$$AD = AC - DC = 13 - 12 = 1 \text{ inch.}$$

Therefore the required reaction is 12 lbs. and one inch of the string projects from the wall.

59. **Theorems.**—(1) Three forces proportional in magnitude to the sides of a triangle and bisecting these sides at right angles will be in equilibrium if they act all inwards or all outwards.

For, by Euc. IV. 5, the perpendicular bisectors of the sides all meet in the centre of the circumscribing circle. Hence the forces all pass through one point.

Moreover, if the triangle is turned through a right angle, its sides, taken in order, will be parallel to the forces, and will represent them both in magnitude and direction. Hence the forces are in equilibrium.

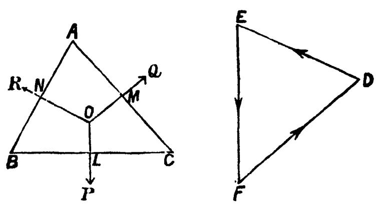


Fig. 56.

(2) Any number of forces proportional to the sides of any polygon and bisecting these sides at right angles will be in equilibrium if they act all inwards or all outwards.

Let $ABCDE...$ be the given polygon. Join AC, AD, ..., and suppose the forces to all act outwards.

From the last theorem, the forces of magnitude AB, BC, bisecting AB, BC at right angles, have a resultant of magnitude AC, bisecting AC at right angles, acting *inwards* with respect to the triangle ABC, or *outwards* with respect to ACD.

Similarly, the forces AC and CD have a resultant AD bisecting AD at right angles.

Finally, the forces AD, DE, EA, bisecting their respective sides at right angles, are in equilibrium.

Therefore the given forces are in equilibrium. The same is similarly true if they act inwards.

(3) To deduce the conditions of equilibrium of three parallel forces acting on a rigid body.*

From Theorem (1), three forces P, Q, R acting perpendicular to the sides of a triangle ABC at their middle points L, M, N will balance if they are proportional to the sides, that is, if

$$\frac{P}{BC} = \frac{Q}{CA} = \frac{R}{AB}.$$

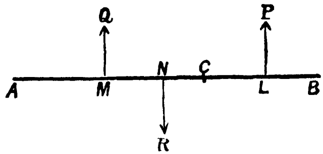


Fig. 57.

This is true however small the altitude of the triangle ABC. Hence it must be true when the altitude is infinitesimal. In this case the points A, B, C lie in one straight line, and the forces P, Q, R act perpendicular to this line, and are therefore parallel.

Now $$AM = \tfrac{1}{2}AC, \quad AN = \tfrac{1}{2}AB;$$
$$\therefore \quad MN = \tfrac{1}{2}CB.$$

Similarly, $$NL = \tfrac{1}{2}AC \quad \text{and} \quad LM = \tfrac{1}{2}BA.$$

Hence the conditions of equilibrium of the three parallel forces P, Q, R acting at L, M, N become

$$\frac{P}{MN} = \frac{Q}{NL} = \frac{R}{LM} \quad \text{.........................} \quad (2).$$

In the above figure $ML = MN + NL$,

and LM is equal and opposite to ML.

Therefore, from (1), R is equal and opposite to $P + Q$, so that, if the directions of forces are denoted by their signs, then, *algebraically*,

$$P + Q + R = 0$$

is a necessary condition of equilibrium.

* The conditions of equilibrium of three or more parallel forces will be treated more fully by other methods in Chap. VI. The present treatment may therefore be omitted on first reading, although it affords an instructive exercise.

Summary of Results.

The Principle of Transmission of Force asserts that the effect of a force acting on a rigid body does not depend on what point in its line of action is its point of application. (§ 52.)

Three Forces in equilibrium must either be parallel or intersect in one point. (§ 53.)

Conditions of equilibrium of a *smooth wedge* under forces P, Q, R perpendicular to faces a, b, c are

$$P/a = Q/b = R/c \ldots\ldots\ldots (1). \quad (\S\, 56.)$$

The weight of a uniform *rod* acts at its *middle point*.

The weight of a sphere, cube, or circle acts at its centre. (§ 57.)

EXAMPLES IV.

1. A person ascends a ladder resting on a rough horizontal floor against a smooth vertical wall; determine the direction and magnitude of the force with which the ladder presses against the floor.

2. A uniform rod BC, 6 ins. long, weighs 2 lbs., and can turn freely about its end B. It is supported by a string AC, 8 ins. in length, attached to a point A in the same horizontal line as B, the distance AB being 10 ins. Find, by a diagram, the tension of AC.

3. AB is a wall, and C a fixed point at a given perpendicular distance from it; a uniform rod of given length is placed on C, with one end against AB. If all the surfaces are smooth, find the position in which the rod is in equilibrium.

4. Two forces applied at two points A, B of a rigid body in the straight line AB are such as to balance one another. Prove (without assuming the Principle of Work) that, when the body moves in the direction AB, the works done by the forces are equal and opposite.

5. A uniform rod, whose length is 8 ft. and weight 16 lbs., is placed over a smooth peg, so that one end rests against a smooth vertical wall. The distance of the peg from the wall is 6 ins. Find the position of equilibrium and the force of pressure on the peg.

6. A uniform rod, whose length is 2 ft. and weight 1 lb., is placed over a smooth peg, so that one end rests against a smooth vertical wall. The thrust on the peg is 8 oz. Find the distance of the peg from the wall and the position of equilibrium.

7. A uniform rod AB, inclined at an angle of 30° to the horizon, rests with the end A in contact with a rough horizontal table, the end B being supported by a string attached to a point C vertically above A. If BC be inclined at an angle of 60° to the horizon, find the reaction of the table and the tension of the string.

8. A rod AB is hinged at A, and supported in a horizontal position by a string BC making an angle of 45° with the rod. The rod has a weight of 10 lbs. suspended from B. Find the tension in the string and the force at the hinge. The weight of the rod may be neglected.

9. A rectangular box, containing a ball of weight W, stands on a horizontal table, and is tilted about one of its lower edges through an angle of 30°. Find the thrusts between the ball and the box.

10. AC and BC are two smooth inclined planes at right angles to one another and intersecting at their lowest point C. A uniform heavy rod AB rests in equilibrium against them. Show that its middle point is vertically above C.

11. Construct a triangle whose sides represent the forces acting on the rod in Q. 10, and calculate the forces of pressure of the rod against the planes, the inclinations of the planes being 30° and 60°, and the weight of the rod 1 lb.

12. A picture is suspended from a nail A by strings $ABOCA$ and OA. The former passes through two smooth rings at B, C, so placed that ABC is an equilateral triangle, and at O, the centre of ABC, it is knotted to the string OA, which is also suspended from A. Given the weight W of the picture, determine the tensions of the strings.

EXAMINATION PAPER II.

1. State and prove the principle of Transmissibility of Force.

2. If three forces in one plane keep a body in equilibrium, show that their lines of action will either meet in a point or be all parallel.

3. If a body is partly supported by resting against a smooth fixed surface which it touches at one point, in what direction does the surface react against the body?

4. Find the ratio of the effort to the weight on an inclined plane when the effort acts parallel to the plane.

5. If a power of 8 lbs. applied parallel to an inclined plane support a weight of 17 lbs., what is the thrust on the plane?

6. A weight rests on a smooth inclined plane. Show that the smallest force which will keep it in equilibrium must act along the plane.

7. Forces of 3, 5, and 7 lbs., respectively, act on a particle at the centre of a circle, towards points on the circumference which divide it into three equal parts. Find the magnitude and direction of the force that will balance them.

8. A heavy uniform beam AB is supported at a point C by the prop CD, its extremity A pressing against a smooth wall EF. Determine the conditions of equilibrium.

9. If a, b, c be the breadths of the three faces of a wedge, supposed forced into a fissure in the usual manner, given the action P against its back a, determine the two reactions Q and R against its two sides b and c.

10. A ladder rests against a smooth wall of a house at a slope to the ground of 45°. Draw a figure showing the directions of the forces acting on the ladder, and prove that the force exerted by the ground is $\sqrt{5}$ times the force exerted by the wall.

PART II.

MOMENTS AND PARALLEL FORCES.

CHAPTER V.

MOMENTS OF FORCES IN ONE PLANE.

60. **Forces tending to produce rotation.**—If a body is attached to a fixed axis or hinge about which it can turn freely, we can set the body in motion by applying to it a force in any direction not passing through the axis. And it is easy to verify by a few simple experiments that, the farther off from the axis the force is applied, the more effect it has in turning the body.

Consider, for example, a door which can turn about its hinge. To open or shut the door, we apply a force to its handle in a direction perpendicular to the plane of the door, and at a considerable distance from the hinge. If we press against the woodwork of the door at a point very near the hinge, we shall have to exert a much greater effort to set the door turning, while if we lean against the edge of the door and push it directly against the hinge, it will not turn at all.

When a wheel is turned by a handle, we apply to it a force perpendicular to the arm of the handle. It is easy to verify that, the further the handle is from the centre of the wheel, the less will be the force required to turn the wheel, although the handle will have to be moved through a greater distance in each revolution.

In this chapter we shall consider the equilibrium of bodies under forces which tend to turn them about a fixed point or axis; but shall not consider the actual motion of such bodies when the forces cause them to rotate.

61. DEFINITION.—*The* **moment of a force about a fixed point** *is the product of the measure of the force into the perpendicular distance of the point from its line of action**.

Thus, if P be the force, and if OM is drawn from any point O perpendicular on the line of action of P, the product $P \times OM$ is **called** the **moment** of the force P about O.

The length OM may be called the **arm** of the moment. Therefore

moment = force × arm.

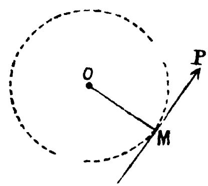


Fig. 58.

The product $P \times OM$ becomes zero if either of its factors diminish to zero, that is, if

$$P = 0 \quad \text{or} \quad OM = 0.$$

In the latter case O is on the line of action of P.

Hence the moment of a force about a point vanishes when either—

(i.) the force itself is zero;

(ii.) the line of action of the force passes through the point.

If the body is fixed or hinged so that it can turn perfectly freely about O, the force P cannot set the body in motion if its line of action passes through O, for by the Principle of Transmission of Force P may be supposed to be applied at O, and this point is prevented from moving by the hinge or support.

Hence, *when the moment of a force about a fixed point vanishes, the force has no tendency to turn the body about that point.*

On the other hand, if the moment is not zero, the force P cannot pass through O. But the reaction of the hinge or support passes through O. Hence these two forces cannot act in the same straight line, and the body cannot remain in equilibrium. It must therefore turn about O, since no other motion is possible.

* Notice that the words *moment and momentum* have entirely different meanings. *They must be carefully distinguished, for no connexion whatever exists between them.*

From these and other considerations to be proved shortly, it may be inferred that the moment of a force about any point is a proper measure of the tendency of the force to produce rotation about that point.

62. Positive and negative moments.—In dealing with forces in one plane, it is convenient to regard the moment of a force about a point as positive or negative according to which way the force tends to rotate a body about that point.

Moments which tend to produce rotation in the *opposite* direction to that in which the hands of a watch go are considered **positive.**

Moments which tend to produce rotation in the *same* direction as the hands of a watch go are therefore to be regarded as **negative**, and a minus sign is prefixed to their amount.

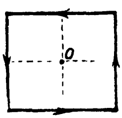


Fig. 59.

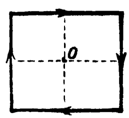


Fig. 60.

Thus, in Figs. 58, 59, the moments about O of the forces indicated by the arrows are all positive; but in Fig. 60 they are all negative.

The following rule is also convenient—

The moment of a force is

positive about all points on the **left** of its line of action,
negative ,, ,, ,, **right** ,, ,,
as seen by a person looking in the direction of the force.

In forming the **algebraic sum of the moments** of any number of forces, each moment is taken with its proper algebraic sign.

Thus, if two forces have equal moments about a point, but tend to produce rotation in opposite directions, their algebraic sum is zero, for one moment is a *plus* and the other a *minus* quantity. In particular, two equal and parallel forces tending in the same direction will have equal and opposite moments about a point midway between them, and the algebraic sum of the moments will be zero.

63. Geometrical representation of the moment of a force.

If a force be completely represented by a straight line, its moment about any point shall be measured by twice the area of the triangle which the straight line subtends at that point.

Let AB represent any force P, then shall the moment of P about O be represented by twice the area OAB.

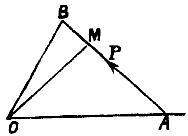


Fig. 61.

Draw OM perpendicular to AB. Then

$$\text{area of } \triangle OAB = \tfrac{1}{2}AB \times OM.$$

(Appendix on Trigonometry and Mensuration, § 20.)

Now AB represents the force P, therefore AB must contain P units of length. Hence

$$\text{moment of } P \text{ about } O = P \times OM = AB \times OM = 2\triangle OAB.$$

64. **If a given force is compounded with any force which passes through O, the moment about O will be unaltered.**

For if AB represents the given force P, and AD represents any force Q passing through O, their resultant R will be represented by AC the diagonal of the parallelogram $ABCD$. Since OAD and BC are parallel,

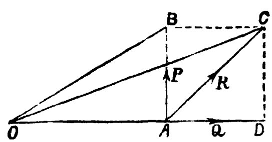


Fig. 62.

$$\therefore \quad \triangle OAB = \triangle OAC;$$

$\therefore$ moment of P about O = moment of R about O; as was to be proved.

65. If a force P is applied at any point A, its moment about any point O is equal to the product

$$OA \times \textbf{resolved part of } P \textbf{ perpendicular to } OA.$$

For let AC represent the force P. Complete the rectangle $ABCD$, whose side AD passes through O. Then, by the Parallelogram of Forces, AD, AB represent the components of P along and perpendicular to OA.

Now $\triangle OAB = \triangle OAC$ (since they are on the same base and between the same parallels).

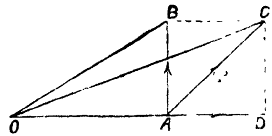


Fig. 63.

Therefore moment of P about O

$$= 2\triangle OAC = 2\triangle OAB$$
$$= OA \times AB$$
$$= OA \times \text{resolved part of } P \text{ perpendicular to } OA.$$

66. **Another expression for the moment of a force.**— In the right-angled triangle OAM (Fig. 63) we have

$$\sin MAO = \frac{MO}{AO} = \frac{OM}{OA}; \quad \therefore \quad OM = OA \sin MAO;$$

$$\therefore \quad \text{moment of } P \text{ about } O = P \,.\, OM = P \,.\, OA \,.\, \sin MAO.$$

Hence the moment of a force about a point is the product of the force, the distance of its point of application from the point, and the sine of the angle which this distance makes with the line of action of the force.

Cor. Since $P \sin MAO$ = resolved part of P perpendicular to OA, we have an independent proof of the property proved in § 65.

67. **The moments of two intersecting forces about any point in the line of action of their resultant are equal and opposite.**

Let two forces P, Q act in the lines AB, AD, and let C be any point in the line of action of their resultant R.

Complete the parallelogram $ABCD$.

*Choose the scale of representation such that AC represents the resultant force R.**

Then, by the Parallelogram of Forces, AB and AD represent the components P, Q.

Since $ABCD$ is a parallelogram,

$\therefore \quad \triangle CAB = \triangle CDA.$

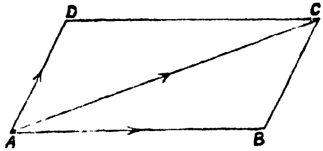


Fig. 64.

But the triangles CAB, CAD represent the moments of P and Q about C, and these moments tend to turn about C in opposite directions.

Therefore the moments of P, Q about C are equal and opposite.

68. **Equilibrium about a fixed point.**—If a rigid body, moveable about a fixed point, is acted on by two forces in any plane through that point, these forces will balance if their moments about that point are equal and opposite.

For, in order that the forces may balance, their resultant must pass through the fixed point. Hence the moments of the forces about that point must be equal and opposite, from § 67.

Either of the forces, if it were to act alone, would set the body turning round about the fixed point. Hence, since the body does not turn when both act, we are led to infer that the tendencies of the forces to produce rotation are equal and opposite.

Hence equal moments about a point represent equal tendencies to produce rotation about that point. Moreover, if a force be doubled, its moment about any point is also doubled; but it is natural to suppose that its tendency to produce rotation about any point is doubled. Hence we infer that the moment of a force about any point is a measure of its tendency to produce rotation about that point.

If two or more forces acting on any body have a single resultant, we should naturally expect the resultant to have the same tendency to produce rotation about any point as the several forces acting simultaneously. We shall now prove that such is the case.

* This step of the proof should be carefully noted, as it is most important.

69. The algebraic sum of the moments of two forces about a point in their plane is equal to the moment of their resultant about that point (Varignon's Theorem).

Let two forces P, Q be completely represented by the lines AB, AD. Let the parallelogram $ABCD$ be completed so that AC represents the resultant of P, Q, and let this resultant be R.

Let O be any point in the plane of the forces. Then

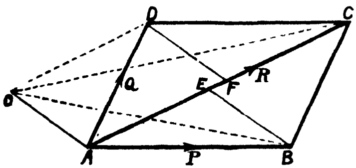


Fig. 65.

their moments about O are measured by twice the triangles OAB, OAD, OAC respectively.

Draw BE, DF parallel to OA, cutting AC in E, F.

Then, evidently, the triangles BCE, DAF are equal in all respects, and therefore $AF = EC$.

Case i.—If O lies *without* the angle BAD, as in Fig. 65, then we have to show that

$$2\triangle OAB + 2\triangle OAD = 2\triangle OAC.$$

Now* $\triangle OAB = \triangle OAE$ ($\because$ EB is parallel to OA);

$\triangle OAD = \triangle OAF = \triangle OEC$ ($\because$ bases AF, EC are equal);

$\therefore\ 2\triangle OAB + 2\triangle OAD = 2\triangle OAE + 2\triangle OEC = 2\triangle OAC$,

or moment of P + moment of Q = moment of force R.

* To avoid complicating the figure, the triangles OAE, OAF are not completed. The student should draw fresh figures on a large scale, filling in these triangles as they are required in the proof.

CASE ii.—If O lies *within* the angle CAD, as in Fig. 66, the moment of Q is negative and is represented by *minus*

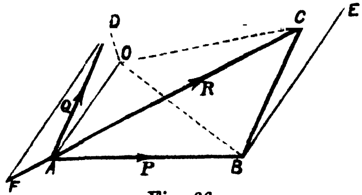


Fig. 66.

twice the area ODA, and we have to show that

$$2\triangle OAB - 2\triangle ODA = 2\triangle OAC.$$

Now, as before, $\triangle OAB = \triangle OAE,$

$$\triangle ODA = \triangle OFA = \triangle OCE;$$

$\therefore\quad 2\triangle OAB - 2\triangle ODA = 2\triangle OAE - 2\triangle OCE = 2\triangle OAC;$

or moment of P + moment of Q = moment of force R, as before.

70. OBSERVATIONS.—The student should write out full proofs of the theorem, with figures, suited to the following cases : — (*a*) When O lies within the angle BAC. (*b*) When O lies within the vertically opposite angle to BAD formed by producing BA, DA.

The proof may be simplified by making one of the sides BC or CD of the parallelogram $ABCD$ pass through O. To do this we draw OD parallel to AB, and *choose the scale of representation such that AD represents the force Q.* Let AB represent P on this scale.

Completing the parallelogram $ABCD$, the resultant R is represented by AC. Now $AB = DC$, $\therefore\ \triangle OAB = \triangle DAC$;
and if O lies without the $\angle BAD$, then

$$\triangle OAD + \triangle OAB = \triangle OAD + \triangle DAC = \triangle OAC,$$

or sum of moments of P, Q = moment of R.

The construction of the figure and completion of the proof are left as an exercise to the student.

[The more advanced student will find that the proof of Case i. may be made to include every case by making the following convention as to the sign of the area of a triangle, *viz.*, that the area is to be regarded as *positive* or *negative* according to whether in going round the sides in the order of the letters used in naming the triangle we always have the triangle on our *left* or *right*. With this convention, $\triangle ODA$ will represent an area equal and opposite to $\triangle OAD$, so that $\triangle ODA + \triangle OAD = 0$, and the proofs of the two cases given above will be found to be identical.]

71. **Generalization.**—*If any number of forces act at a point in one plane, the algebraic sum of their moments about any point in that plane is equal to the moment of their resultant.*

Let P_1, P_2, P_3 be any number of forces acting in one plane at A. Let O be any point in that plane.

Let X_1, Y_1 be the resolved part of P_1 along and perpendicular to OA. Let X_2, Y_2 and X_3, Y_3 be the resolved parts of P_2, P_3. Then

$$\text{moment of } P_1 \text{ about } O = \text{moment of } Y_1 \text{ about } O = OA \times Y_1,$$

and similarly for the others.

Therefore

$$\text{algebraic sum of moments of forces about } O = OA \times (Y_1 + Y_2 + Y_3 + \ldots).$$

But, if R denotes the resultant, X, Y its resolved parts along and perpendicular to OA, then

$$Y = Y_1 + Y_2 + + Y_3 + \ldots; \qquad (\S\ 30.)$$

$$\therefore \quad \text{algebraic sum of moments about } O = OA \times Y$$
$$= \text{moment of } R \text{ about } O.$$

The same thing may also be proved by combining two of the forces and then combining their resultant with a third force, and so on.

Cor. 1. If any number of forces acting at a point in one plane are in equilibrium, the algebraic sum of their moments about any point in the plane is zero.

For their resultant is zero; therefore its moment about any point is zero.

Cor. 2. If the algebraic sum of the moments of any number of forces about a point O in their plane vanishes, the forces *either* are in equilibrium *or* have a resultant passing through O.

72. **Difference between the moment and the work of a force.**—If we compare the definitions of § 36 and § 61, we shall observe that both the moment of a force about a point and the work of a force are products of forces into lengths of straight lines. But the moment of a force about a point is the product of a force into a straight line **at right angles to it,** while work is the product of a force into a straight line **in the same direction.**

Thus, let a force P be applied to a particle at A. Then

moment of P about O

$= P \times OM$ (Fig. 67)

$= OA \times$ resolved part of P *perpendicular* to OA;

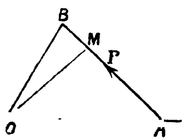


Fig. 67.

work done in moving the particle from A to O

$$= P \times AM = AO \times \text{resolved part of } P \text{ along } AO.$$

It is also to be observed that the moment of a force is a purely statical idea, but *work is only done when motion takes place.*

*73. We may apply the Principle of Work to show that the moment of a force about any point measures its tendency to produce rotation about that point, as follows :—

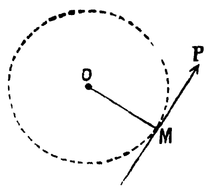


Fig. 68.

Let a body be rotated about the fixed point O by a force of magnitude P, applied at the end of the arm OM and *always acting perpendicular to OM*. Then, in the course of one complete turn about O, the particle M describes a circle whose centre is O and whose circumference is $2\pi \times OM$. But the force P always acts in the direction in which M is moving. Hence the work done by P is found by multiplying P into the whole length of the path described by M;

$$\therefore \quad \text{work of } P \text{ in one revolution} = P \times 2\pi . OM.$$

$$= 2\pi \times \text{moment of } P \text{ about } O.$$

But the work done in one turn must evidently be proportional to the tendency of P to produce rotation. Therefore this tendency is proportional to the product $P \times OM$; that is, to the moment of P about O.

Again, suppose a body, moveable about a fixed point O, is acted on by any number of forces in equilibrium. Let it be rotated through one complete turn, and let the directions of the forces rotate with the body. Then the forces will continue to remain in equilibrium; therefore, by the Principle of Work, the sum of the works done by the several forces is zero, *i.e.*,

$$\text{sum of moments of forces} \times 2\pi = 0.$$

Therefore the algebraic sum of the moments of the forces about O must be zero; thus affording an alternative proof of § 71, Cor. 1.

SUMMARY OF RESULTS.

Moment of force P about point O

$= P \times$ (perpendicular distance of P from O). (§ 61.)

Moments are *positive* which tend in *opposite* direction to hands of clock. (§ 62.)

Moments are positive about points on *left-hand* of force.

If force P is represented by AB, then its moment about O

$= 2 . \Delta OAB$. (§ 63.)

$= OA \times$ (resolved part of P *perpendicular* to OA). (§ 65.)

Varignon's Theorem or the Equation of Moments.—The moment of resultant of two (or more) forces = algebraic sum of moments of its components. (§ 69.)

[Proved for parallel forces in the next chapter, § 79.]

For equilibrium about a fixed point, the algebraic sum of the moments must be zero. (§ 68 and § 71, Cor. 2.)

EXAMPLES V.

1. Find the moment round a point O of a force of 3 lbs. acting at a point A along a line AB, where OA is equal to 10 ins., and the angle OAB equals (i.) 90°, (ii.) 45°, (iii.) 135°, (iv.) 120°.

2. ABC is a triangle, right-angled at C, and having the angle B equal to 60°. Forces 3, 4, 5 act along AB, BC, CA, respectively. Find the moment of each round the opposite angular point.

3. OA, OB are chords, 4 and 5 ins. in length, of a circular disc $OACB$, whose diameter OC is 6 ins. If forces of 3 and 4 lbs. act from O along these chords respectively, find how the disc will begin to move, the point C being fixed.

4. Explain why a boy, pulling hold of the rim of a garden roller at its top point, can pull back a much stronger boy catching hold of the handle. If the first boy pulls the roller back in a horizontal direction, and the second pulls the handle in a direction making an angle of 30° with the horizon, compare the forces which they exert on it if no motion takes place.

5. Forces act along all the sides but one of a plane polygon, and are represented by the sides in magnitude; prove that, when they all act the same way round, their resultant is a force parallel to the remaining side and represented by it in magnitude. Prove, also, that

the line of this resultant forms with the remaining side a parallelogram whose area is twice the area of the polygon.

6. A horizontal rod 8 ft. long has a weight of 1 lb. at one end, upward forces of 2 and 3 lbs. act at distances 2 and 6 ft. from that end, and a weight of 4 lbs. hangs from the other end. Taking the first end as the left-hand end in a figure, write down the moments of the forces about each end of the rod and about its middle point, prefixing the proper sign to each. Also find the algebraic sums of the moments about these points.

7. Forces of 1, 3, 5, 7, 9, 11 lbs. act along the sides AB, BC, &c., of a regular hexagon $ABCDEF$. Find the moment of each force about the point A.

8. Draw an equilateral triangle ABC, and suppose each side to be 4 ft. long; a force of 8 units acts from A to B, and a force of 10 units from C to A. (*a*) Find the moment of each force with reference to the middle point of BC. (*b*) Find a point with reference to which the forces have equal moments of opposite signs.

9. Draw a square $ABCD$; suppose forces of P and Q units to act from A to B and from A to D, respectively. Find the perpendicular distance of the line of action of their resultant from C.

10. Two forces of 5 lbs. and 12 lbs., respectively, act at right angles. Find the locus of the points in their plane round which the sum of their moments is $3\frac{1}{4}$. (*Apply Varignon's Theorem.*)

11. A system of forces in a plane is such that the sum of their moments about a point A in that plane vanishes. If the forces are not in equilibrium, what do we know about their resultant?

12. A square whose side is 3 ft. long is divided into 9 smaller squares, each 1 ft. square. Forces of 3, 4, 5, 6 lbs. act along the sides of the middle square taken in order, in such directions that their moments about any point inside that square are all positive. Write down the moments of each force about each of the four angular points of the large square, prefixing the proper sign to each, and find the algebraic sum of the moments about each of these four points.

13. $ABCD$ is a square. Equal forces (P) act from D to A, A to B, and B to C, respectively, and a fourth force $2P$ acts from C to D. Find a point such that, if the moments of the forces are taken with respect to it, the algebraical sum is zero.

CHAPTER VI.

PARALLEL FORCES.

74. **"Like" and "unlike" parallel forces.**—In the preceding chapters we have considered the equilibrium of forces whose directions intersect one another. In the practical applications of mechanics, however, parallel forces are of even more frequent occurrence than intersecting forces.

The following definitions will be required :—

Definition.—Parallel forces which tend in the same direction are said to be **like.** Those which tend in opposite directions are said to be **unlike.**

If a force acting in one direction be regarded as *positive*, it is convenient to regard any unlike force as *negative*.

Thus, if there be forces of 28 lbs., acting upwards, and 56 lbs., acting downwards, and we consider the upward direction as positive, the two forces will be unlike, and the complete expressions for them will be $+28$ lbs. and -56 lbs., respectively.

75. **Composition of parallel forces (preliminary observations).**

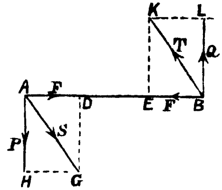


Fig. 69.

When two parallel forces P, Q act at any two points A, B of a rigid body, their resultant, if it exists, may be deduced from the resultant of two intersecting forces by making use of the theorem of § 51. If, at A and B, we apply to the body equal

and opposite forces F and $-F$, both acting in the straight line AB, these will not affect the equilibrium of the body as a whole. Now compound P with F, and Q with $-F$.

The two former are equivalent to a single force S at A, and the two latter to a single force T at B; hence the two forces S, T are equivalent to P, Q.

Now let the forces S, T, on being produced, intersect one another in O. Then they must be equivalent to a single resultant force passing through O. This force will also be the resultant of the parallel forces P, Q.

Hence P, Q are equivalent to a single resultant force.

The method fails if the forces S, T are themselves parallel, and we shall now show that *this only happens when P, Q are equal unlike forces.*

For if $ADGH$, $BEKL$ be the Parallelograms of Forces at A, B, respectively,

then, since DG is parallel to KE and AG to BL,

$$\therefore \quad \angle ADG = \angle BEK, \quad \text{and} \quad \angle DAG = \angle EBK.$$

Also, since AD, BE represent equal and opposite forces,

$$AD = EB.$$

$$\therefore \quad DG = KE, \quad \text{or} \quad AH = LB.$$

Therefore the forces P, Q, represented by AH, BK, are equal and opposite.

76. **Couples.** — Definition. — Two equal but unlike parallel forces are said to constitute a **couple.***

Hence the result just proved shows that *two parallel forces are always equivalent to a single resultant force unless they constitute a couple.* By § 51 the two forces forming a couple are not in equilibrium.

In the present chapter we shall deal with parallel forces that do not form couples. We shall, therefore, always assume that they have a resultant or else that three or more of them are in equilibrium.

* Hence a *couple* in Statics does *not* simply mean "two of anything" as in ordinary language.

77. To find the resultant of two *like* parallel forces.

Let P, Q be the two parallel forces, A, B any two points on their lines of action which we may suppose to be their points of application.

At A introduce a force of magnitude F along AB, and at B introduce an equal and opposite force F along BA (that is, $-F$ along AB); these two forces will balance

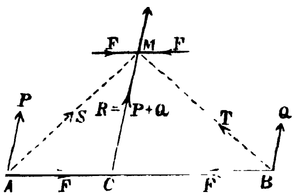


Fig. 70.

each other, and will not affect the resultant. Let S be the force compounded of P and F, T the force compounded of Q and $-F$.

Then the forces S and T will intersect at a point M between the forces P, Q, and will have a resultant R acting at M, which will also be the resultant of P and Q.

Draw CM parallel to P or Q, cutting AB in C.

Replace the force S at M by its components along CM, and parallel to AC. These components are, of course, the same as when S acted at A, and are therefore P, F. Similarly, replace the force T at M by its components along CM and parallel to AC; these components are Q, $-F$.

Thus the two forces P, Q or S, T are together equivalent to forces $P+Q$ acting along CM, and $F-F$ or zero parallel to AC.

Hence the resultant of P, Q is $P+Q$ acting along CM; or, in words—

The *magnitude* of the resultant is the sum of the components.

The *direction* of the resultant is parallel to the components.

The *position* of the resultant may be found as follows :—

Since the forces at A are parallel to the sides of $\triangle ACM$, therefore, by the Triangle of Forces,

$$\frac{P}{F}=\frac{CM}{AC}, \quad \text{or} \quad P\times AC = F\times CM.$$

Similarly,* since the forces at B are parallel to the sides of $\triangle BCM$,

$$\frac{Q}{F}=\frac{CM}{CB}, \quad \text{or} \quad P\times CB = F\times CM.$$

$$\therefore \quad P\times AC = Q\times CB.$$

If AB is taken at right angles to the forces, this relation expresses the fact that **the moments of P, Q about C are equal and opposite.**

We may also write the above relation,

$$\frac{AC}{CB}=\frac{Q}{P};$$

which shows that AC is greater or less than CB, according as Q is greater or less than P. Hence

The resultant lies between the two forces P, Q, and is nearest to the greater force.

It divides the line AB in the inverse proportion of the forces.

---

* The sides of $\triangle ACM$ do *not* represent the forces at A *on the same scale* that the sides of $\triangle BCM$ represent the forces at B. In the former triangle, AC represents F and CM represents P. In the latter, BC represents F and CM represents Q. Hence *the scales of representation in the two triangles are different.*

Note also that, *algebraically*, we should strictly have $Q/(-F) = CM/BC$, and, since $CB = -BC$, this gives $Q/F = CM/CB$, as stated.

line perpendicular to the forces P, Q, R and cutting them in A, B, C, respectively. In this straight line apply two equal and opposite forces F, $-F$. Let S, T be the forces

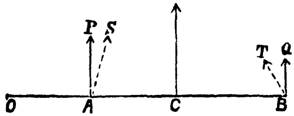


Fig. 72.

compounded of P, F and Q, $-F$, respectively; then R, the resultant of parallel forces P, Q, is also the resultant of the forces S, T. Now the forces S, T intersect one another (since P, Q do not form a couple).

$\therefore$ moment of R about O = algebraic sum of moments of S, T.

Now moment of R about $O = R \times OC$.

Also P, Q are the resolved parts of S, T perpendicular to $OABC$.

$\therefore$ moment of S about $O = P \times OA$;

moment of T about $O = Q \times OB$.

$$\therefore \quad R \times OC = P \times OA + Q \times OB;$$

or moment of R about O = algebraic sum of moments of P, Q about O.

This relation is sometimes called the **Equation of Moments.**

COR. 1. When three parallel forces are in equilibrium, the algebraic sum of their moments about any point in their plane is zero.

For each force is equal and opposite to the resultant of the other two; therefore its moment is equal and opposite to the sum of their moments.

COR. 2.—When any number of coplanar forces* act on a rigid body, the algebraic sum of their moments about any point in their plane is equal to the moment of their resultant.

Consider two of the forces. Whether these intersect or are parallel, algebraic sum of their moments about O = moment of their resultant.

Combine this resultant with a third force; then, algebraically, algebraic sum of moments of the three

= moment of third force + moment of resultant of first two

= moment of resultant of all three forces,

and so on, till all the forces have been compounded together.

OBSERVATION.—In finding the position of the resultant of two or more parallel forces, it is usually better to apply the present principle by "taking moments," *i.e.*, writing down the "equation of moments" about any convenient point, instead of employing the results proved in § 77 or § 78. As a rule it is most convenient to take a point in the line of action of one of the forces, because the moment of the force is then zero. We may, however, take moments about a point on the resultant, or about any point whatever.

Examples.—(1) To find the resultant of two like forces of 5 lbs. and 4 lbs. applied at the ends of a rod 3 ft. long perpendicular to its length.

Let AB be the rod, C the point at which the resultant cuts it. (Fig. 72 may be used, taking $P = 5$ lbs. and $Q = 4$ lbs.)

Since the forces are like, their resultant $= 4 + 5$ lbs. $= 9$ lbs.

Take moments about A. Then the moment of 9 lbs. at C equals the sum of the moments of 5 lbs. at A and 4 lbs. at B, whence, taking a foot as the unit of length,

$$9 \times AC = 5 \times 0 + 4 \times AB = 0 + 4 \times 3 = 12 \cdot$$

$$\therefore \quad AC = \tfrac{12}{9} \text{ ft.} = 1\tfrac{1}{3} \text{ ft.} = 16 \text{ ins.}$$

Therefore the resultant acts at a distance of 16 ins. from the 5 lb. force.

(2) To find the resultant of two *unlike* forces of 5 lbs. and 4 lbs. applied at points distant 3 ft. apart.

Let A, B, C be the points of application of the forces and their resultant.

Since the forces are unlike, their resultant $= 5 - 4$ lbs. $= 1$ lb.

* *I.e.*, forces in one plane.

Taking moments about A, the equation of moments gives

$$1 \times AC = 5 \times 0 - 4 \times AB = 0 - 4 \times 3 = -12 ;$$
$$\therefore \quad AC = -12 \text{ ft.} \quad \text{or} \quad CA = 12 \text{ ft.}$$
$$\therefore \quad CB = 12 + 3 = 15 \text{ ft.}$$

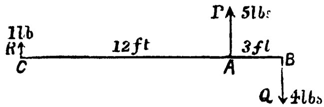


Fig. 73.

Therefore the resultant acts at distances of 12 and 15 ft. from the 5-lb. and 4-lb. forces respectively.

(3) To find parallel forces which must be applied to a bar 9 ft. long in order that their resultant may be a force of 12 lbs. acting at 2 ft. distance from one end.

Let P, Q be the required forces, BC the bar, A the point at which the resultant acts.

Taking moments about C, we have

$$P \times CB = Q \times 0 + 12 \times CA ;$$
$$\therefore \quad P \times 9 = 12 \times 2 = 24 ;$$
$$\therefore \quad P = \tfrac{24}{9} \text{ lbs.} = \tfrac{8}{3} \text{ lbs.} = 2\tfrac{2}{3} \text{ lbs.}$$
$$\text{Also} \quad P + Q = 12 \text{ lbs.} ;$$
$$\therefore \quad Q = 12 - 2\tfrac{2}{3} = 9\tfrac{1}{3} \text{ lbs.}$$

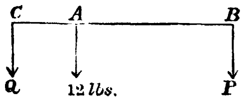


Fig. 74.

80. Conditions of equilibrium of three parallel forces.

If three parallel forces are in equilibrium, each force is proportional to the distance between the other two.*

Let P, Q, R represent the forces algebraically (so that P, Q, R are positive or negative according to the directions of the forces). Let a straight line be drawn, cutting them at right angles in A, B, C.

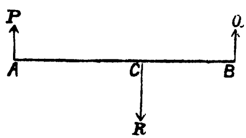


Fig. 75.

Then the sum of the moments of P, Q, R about any point is zero.

* For the following proof § 81 may be substituted as an alternative.

Taking moments about A, B, C in succession, we have

$$P.0 \ + Q.AB + R.AC = 0;$$
$$P.BA + Q.0 \ + R.BC = 0;$$
$$P.CA + Q.CB + R.0 \ = 0.$$

These give, respectively,

$$\frac{R}{AB} = -\frac{Q}{AC}; \quad \frac{P}{BC} = -\frac{R}{BA}; \quad \frac{Q}{CA} = -\frac{P}{CB}.$$

Now CA is equal and opposite to CA—that is,

$$AC = -CA.$$

Similarly, $BA = -AB$, $CB = -BC$.

With these substitutions, the last relations become

$$\frac{R}{AB} = \frac{Q}{CA}; \quad \frac{P}{BC} = \frac{R}{AB}; \quad \frac{Q}{CA} = \frac{P}{BC}.$$

Hence the conditions of equilibrium may be written in the symmetric form,

$$\frac{P}{BC} = \frac{Q}{CA} = \frac{R}{AB} \quad \text{.................. (1).}$$

[Notice that P stands over the length which does *not* contain the letter A; Q, R over the lengths which do not contain B, C, respectively.]

This relation holds good for the directions as well as the magnitudes of the forces.

Thus, taking the points in the order of Fig. 75, since A, B, C lie in a straight line, the two lengths BC, CA are together equal and opposite to the third length AB, so that, algebraically, $BC + CA + AB = 0$.

Hence the two forces P, Q are together equal and opposite to the third, so that, algebraically,

$$P + Q = -R, \quad \text{or} \quad P + Q + R = 0$$

The conditions of equilibrium of three parallel forces may also be stated thus:

The two extreme forces act in the same direction, and the middle force acts in the reverse direction and is equal and opposite to their sum. Also each force is proportional to the distance between the other two.

Example.—A rod 10 ft. long, whose weight W acts at its middle point, has a weight $3W$ attached to one end. To find at what point it must be supported in order to rest balanced.

Let A be the end, C the middle point of the rod, D the point of support, and let R denote the reaction at D.

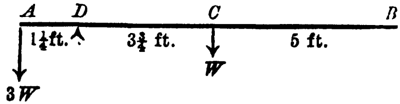


Fig. 76.

Then, by the conditions of equilibrium,

$$\frac{W}{AD} = \frac{3W}{DC} = \frac{R}{CA};$$

$$\therefore \quad DC = 3AD;$$

$$\therefore \quad AC = AD + DC = 4AD \quad \text{and} \quad AD = \tfrac{1}{4}AC.$$

But $AC = 5$ ft. Therefore

$$AD = \tfrac{5}{4} \text{ ft.} = 1\tfrac{1}{4} \text{ ft.}, \quad DC = 3AD = 3\tfrac{3}{4} \text{ ft.}$$

Hence the rod balances about a point $1\frac{1}{4}$ ft. from the end at which $3W$ is attached.

81. **Generalization.**—If the straight line ABC, instead of being drawn perpendicular to the forces, is drawn *cutting them in any other direction*, the conditions of equilibrium are still expressible in the form,

$$\frac{P}{BC} = \frac{Q}{CA} = \frac{R}{AB}.$$

For, since R is equal and opposite to the resultant of P and Q, we have, by § 77 or § 78,

$$Q = P \times \frac{CA}{BC} \text{ and } R = -(P+Q) = -P\left(1 + \frac{CA}{BC}\right)$$

$$= -P\frac{BC + CA}{BC} = -P\frac{BA}{BC} = P\frac{AB}{BC};$$

whence

$$\frac{P}{BC} = \frac{Q}{CA} = \frac{R}{AB}.$$

82. Experimental verification. — (*a*) DETAILS OF EXPERIMENT. — To verify, experimentally, the conditions of equilibrium of three parallel forces, take any rod, and first find C the centre of gravity (*i.e.*, the point at which its weight acts) by making it balance on a support placed at that point. Now suspend the rod from two spring balances attached at any two points A, B, and hanging vertically; at C attach any known weight, and read off the spring balances.

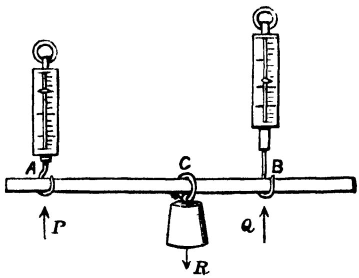


Fig. 77.

(*b*) OBSERVED FACTS. — It will be found that the two readings, when added together, are exactly equal to the weight of the rod together with its attached weight, both of which act at C.

Now let the distances AC, CB be measured. Then, if the experiment be carefully performed, it will be found that the readings of the two balances attached at A, B, and the total weight at C, are proportional, respectively, to the lengths BC, CA, AB.

(*c*) DEDUCTIONS.—Hence, if R denote the total weight at C, and P, Q denote the upward thrusts exerted on the rod by the two spring balances, we shall have

$$R = -(P+Q), \quad P \times AC = Q \times CB,$$

and
$$\frac{P}{BC} = \frac{Q}{CA} = \frac{R}{AB};$$

agreeing with the conditions of equilibrium already found.

Observation.—The experiment can be varied by attaching a third spring balance instead of the weight at C. If the three spring balances are placed horizontally instead of hanging vertically, the weight of the rod will not affect the equilibrium, and it will be found that the readings of the three spring balances attached to A, B, C are respectively proportional to BC, CA, AB, thus verifying the conditions of § 80.

Summary of Results.

The resultant of two like parallel forces P, Q at A, B has its

magnitude $= P + Q =$ sum of forces,

direction parallel to P, Q,

point of application C *between* A, B, such that

$$P \times AC = Q \times CB. \qquad (\S\ 77.)$$

The resultant of two unequal unlike parallel forces P, Q, where $P > Q$, has its

magnitude $= P - Q =$ difference of forces,

direction parallel to and in sense of greater force P,

point of application C on the side of the greater force, remote from the lesser (*i.e.*, on BA produced through A), so that

$$P \times CA = Q \times CB. \qquad (\S\ 78.)$$

The conditions of equilibrium of three parallel forces P, Q, R acting on a rod at A, B, C may be written

$$\frac{P}{BC} = \frac{Q}{CA} = \frac{R}{AB} \quad \dots\dots\dots\dots\dots\dots \quad (1),$$

and the *middle* force is opposite in direction to the other two, and equal to their sum. (§ 80.)

Varignon's Theorem holds good for parallel as well as for intersecting forces, and asserts that the algebraic sum of the moments of two or more forces is equal to the moment of their resultant. (§ 79.)

EXAMPLES VI.

1. Find the resultants of each of the following pairs of like parallel forces at the given distances apart, stating in each case the magnitude of this resultant and its distances from each of the two components, and illustrating by figures :—

(i.) 1 lb. and 3 lbs., 2 ft. apart ,
(ii.) 5 lbs. and 7 lbs., 3 ft. apart ;
(iii.) $10\frac{1}{2}$ lbs. and $1\frac{1}{2}$ lbs., 1 yd. apart.
(iv.) 4 lbs. and 10 lbs., 42 ins. apart :
(v.) $\frac{1}{2}$ ton and $\frac{1}{4}$ ton, 6 ins. apart ;
(vi.) 400 and 600 grammes, 10 cm. apart.

2. Find, in like manner, the resultants of pairs of *unlike* parallel forces whose magnitudes and distances apart are given by the data of Example 1.

3. A rod 10 ft. long, whose weight may be neglected, has masses of 8 lbs. and 11 lbs. attached one to each end. Find the point about which it will balance and the force of pressure on its support.

4. Two unlike parallel forces, of 1 and 2 units respectively, act upon a rigid body, at points 1 ft. apart. Find the magnitude and point of application of their resultant.

5. A uniform bar, 10 ft. long, balances over a rail, with a boy, weighing three times as much as the bar, hanging on to the extreme end of it. Draw a figure showing its balancing position.

6. Show that the resultant of two unlike parallel forces acts towards the side of the greater of the two forces, and can never act between them. What happens if the forces become equal?

7. Two men are carrying a bar 16 ft. long, and weighing 150 lbs. One man supports it at a distance of 2 ft. from one end, and the other man at a distance of 3 ft. from the other end. What weight does each man bear?

8. A uniform rod, 6 ft. long, and weighing 5 lbs., is laid on a table with 6 ins. projecting over the edge. What weight can be hung on the end of the rod before the rod will be pulled over?

9. Two parallel forces P and Q act at two points in a straight line, 6 ins. apart, in opposite directions. Their resultant is a force of 1 lb. acting at a point in the line 4 ft. from the larger of the forces P and Q. Determine the values of P and Q.

10. A uniform heavy beam of length 7 ft. rests horizontally on two supports, one at one end and the other $5\frac{1}{2}$ ft. from that end. If the greatest mass that can be hung on the other end of the beam without disturbing the equilibrium be 16 lbs., find the weight of the beam.

11. Apply the Equation of Moments to deduce the magnitude and position of the resultant of (i.) two like, (ii.) two unlike, parallel forces applied perpendicular to the straight line joining their points of application.

12. Conversely, employ the results proved in §§ 77, 78 to establish the Equation of Moments for two parallel forces.

13. A rod 12 ft. long, whose weight may be neglected, rests horizontally with one end on the edge of a table and the other supported by a vertical string. A mass of 18 lbs. is attached to the rod at a certain point. If the tension of the string be equal to the weight of 12 lbs., find the force of pressure on the table and the point where the weight is attached.

14. Find by how much the greater of two parallel forces P, Q acting in opposite directions must be diminished in order that the distance of the line of action of the resultant from that of P may be the same as that of the line of action of the former resultant was from that of Q?

CHAPTER VII.

MACHINES—THE LEVER—THE WHEEL AND AXLE.

83. **A machine** in Mechanics means any contrivance in which a force applied at one point is made to raise a weight or overcome a resisting force acting at another point. The former force is called the **effort** or **power,** the latter the **resistance** or **weight.*** We may suppose, by way of illustration, that the effort is applied by a man who works the machine, and that the resistance is the weight of a heavy body which the man has to lift off the ground.

Machines are used for the following purposes:—

(1) To enable a person to raise weights or overcome resistances so great that the effort he is capable of exerting would be insufficient without the use of a machine.

Example.—A truck drawn up an inclined plane when it is too heavy to be lifted bodily off the ground.

(2) To enable the motion imparted to one point of a machine to produce a much more rapid motion at some other point.

Example.—A bicycle.

(3) To enable the effort to be applied at a more con-

* The terms "power" and "weight" are used in the older books on Mechanics, and still sometimes occur in examination papers; but "power" also signifies "rate of working," such as horse-power, and machines are often used in overcoming resistances other than those due to gravity or "weight."

venient point, or in a more convenient direction, than that in which the resistance acts.

Example.—A poker used to stir the fire to avoid putting our hands into the glowing coals.

In every case a machine must be capable of *moving* the point of application of the resistance, *i.e.*, of *doing work* against the resistance, and the Principle of Work teaches us that, in all cases, an equal amount of work must be done by the effort in moving the machine; in other words, *we must put as much work into the machine as we want to get out of it.*

84. **The mechanical powers.**—The simplest forms of machines are called the **mechanical powers**, and it is usual to distinguish the following six forms of them:—

The inclined plane.	[Chap. III.]
The wedge.	[Chap. IV.]
The lever.	
The wheel and axle, and windlass.	
The pulley and systems of pulleys.	[Chap. VIII.]
The screw.	[Chap. IX.]

In every case we suppose these machines to be devoid of friction, and in Statics we are chiefly concerned with finding the relations between the effort and the resistance, when there is equilibrium.

85. **Mechanical advantage.** — DEFINITION. — The **mechanical advantage** is the number which expresses what multiple the resistance is of the effort, *i.e.*,

$$\textbf{mechanical advantage} = \frac{\textbf{resistance}}{\textbf{effort}}.$$

Consider, for example, a smooth inclined plane at a slope of, say, 1 in 20. By applying a force of 1 cwt. along the plane, it is possible to draw a weight of 20 cwt. or 1 ton up the plane, and if the plane be long enough, this weight may be raised to any desired height. The resistance to be overcome is that due to gravity, viz., the weight of 20 cwt. It is therefore twenty times the effort, and we say that *the mechanical advantage is* 20. Generally for a smooth incline of 1 in n, the mechanical advantage is n if the effort acts up the plane.

86. The lever is a rigid bar capable of turning freely about a fixed point of support. This point is called the **fulcrum.** The *effort* is a force applied at any point of the lever, so as to turn it about the fulcrum, and thus to raise a weight or overcome a resistance applied at any other point. The **arms** of the lever are the portions joining the fulcrum to the points of application of the effort and resistance.

The lever is most often a straight rod, the two arms therefore being in one straight line, and the effort and resistance generally act perpendicular to the arms. But these are mere matters of convenience.

87. To find the mechanical advantage of the lever when the forces act perpendicular to the arms.— Let C be the fulcrum, CA, CB the arms, and let a force P applied at A perpendicular to CA support a resistance Q applied at B perpendicular to CB. Then the condition of

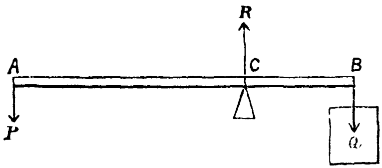


Fig. 78.

equilibrium requires the moments of P and Q about C to be equal and opposite; *i.e.*,

$$P \times CA + Q \times CB = 0, \quad \text{or} \quad P \times CA = Q \times BC$$

(remembering that BC is to be regarded as equal and opposite to CB). This condition may be written

$$\frac{Q}{P} = \frac{CA}{BC}.$$

$\therefore$ **mechanical advantage, or** $\dfrac{\textbf{resistance}}{\textbf{effort}}$,

$$= \frac{\textbf{arm of effort}}{\textbf{arm of resistance}} \quad \textbf{(1)};$$

or *the effort and resistance are inversely proportional to the arms on which they act.*

Hence, by applying the effort at the end of a long arm and the resistance very near the fulcrum, the mechanical advantage may be made very great, and a man may raise a weight many times greater than he could lift bodily off the ground.

Example.—Thus, by exerting a force of 1 lb. at a distance of 2 ft. from the fulcrum, we can lift a weight of 2 lbs. applied at a distance of 1 ft. from the fulcrum, or a weight of 24 lbs. applied an inch away from the fulcrum.

88. **Mechanical advantage of levers in general.**—Where P, Q, the effort and resistance, do not act perpendicularly to CA, CB, the arms of the lever, we must drop

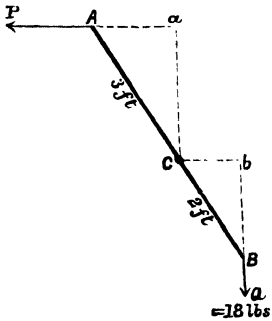


Fig. 79.

Ca, Cb perpendicular on the lines of action of P, Q. By taking moments about C, the condition of equilibrium now becomes $$P \times Ca = Q \times Cb.$$

$\therefore$ **mechanical advantage** $\dfrac{Q}{P}$

$$= \frac{Ca}{Cb} = \frac{\text{perp. dist. of effort from fulcrum}}{\text{perp. dist. of resistance from fulcrum}}.$$

Observations.—The only difference therefore is that the perpendiculars Ca, Cb are no longer the arms of the lever. We might, if

we liked, replace the lever with one whose arms are Ca, Cb without altering the mechanical advantage.

As a rule, in all such cases it is advisable to start with the Equation of Moments in working numerical examples, and not to assume formulæ which are only particular cases of that equation.

Example.—A straight lever, whose arms are 2 ft. and 3 ft. long, rests at an inclination of 60° to the horizon, and a weight of 18 lbs. hangs vertically from its shorter arm. To find the horizontal force which must be applied to its longer arm in order to balance.

Let ACB be the lever (Fig. 79). Then, if P, Q denote the effort and weight, these forces make angles of 60° and 30° respectively with AB; therefore their resolved parts perpendicular to AB are $\frac{1}{2}P\sqrt{3}$ and $\frac{1}{2}Q$ respectively. Their moments about C are the products of these resolved parts into the arms CA, CB. Therefore the Equation of Moments gives

$$\tfrac{1}{2}P\sqrt{3} \times CA = \tfrac{1}{2}Q \times CB;$$

$$\therefore \quad \tfrac{1}{2}P\sqrt{3} \times 3 = \tfrac{1}{2} \,.\, 18 \times 2;$$

$$\text{whence required force } P = \frac{18 \times 2}{3\sqrt{3}}$$

$$= \frac{18 \times 2\sqrt{3}}{9} = 4\sqrt{3} \text{ lbs.}$$

89. The three classes of lever. — *Straight* levers are sometimes divided into three classes, according to the relative positions of A, B, C, the points of application of the effort and resistance and the fulcrum. We shall suppose the effort and resistance to be parallel. In considering the different cases, it is most convenient to use the symmetrical conditions of equilibrium of three parallel forces. If R be the reaction of the fulcrum on the lever, we have, therefore, by § 80,

$$\frac{P}{BC} = \frac{Q}{CA} = \frac{R}{AB}.$$

This relation holds good as regards *both magnitude and direction*, like and unlike forces standing over lengths measured in like and unlike directions, respectively.

The thrust of the lever against its support at C is a force equal and opposite to R. For convenience we will suppose that the resistance Q is a weight which always acts downwards.

90. A lever of the first class (Fig. 80) is one in which the fulcrum is placed between the effort and resistance, the points occurring in the order A, C, B.

Here BC and CA are drawn in one direction, and AB in the opposite direction.

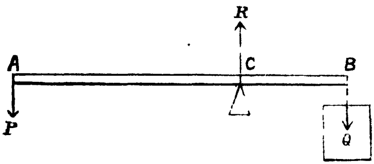


Fig. 80.

Therefore P, Q both act in one direction, and R in the opposite direction.

Thus the effort must be applied *downwards*, and the reaction acts *upwards*, so that the lever presses *downwards* on the fulcrum.

In this lever, BC may either be greater or less than or equal to CA. Therefore the effort may either be greater or less than or equal to the weight which it has to lift; so that *the mechanical advantage may be less, greater than, or equal to unity.*

And since $BA = BC + CA$, the reaction R is equal and opposite to the sum of P and Q, or numerically

$$R = P + Q.$$

Examples of this class.—The handle of a pump; a crow-bar when it rests on a block in front of the weight to be

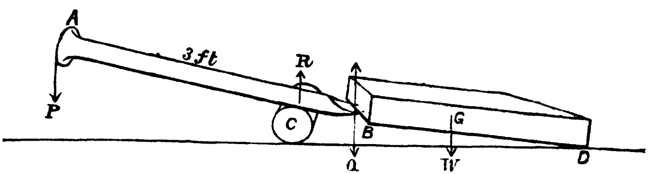


Fig. 81.

lifted and not with its end on the ground (Fig. 81); a

poker, used to raise the coals in a grate, a bar of which is the fulcrum ; a spade, in digging ; a see-saw.

Double lever.—A pair of scissors. Fig. 82 shows the forces acting on the arms of the scissors, those on the dotted arm being accented.

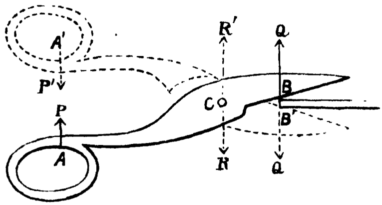


Fig. 82.

91. A lever of the second class (Fig. 83) is one in which the resistance is placed between the fulcrum and the effort, and the points of application, therefore, occur in the order A, B, C.

Hence AB and BC are drawn in one direction, and CA is in the opposite direction.

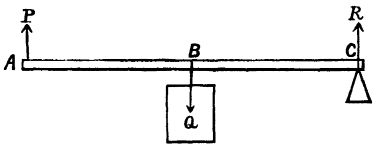


Fig. 83.

Therefore P and R both act in one direction, and Q in the opposite direction.

Thus the effort must be applied *upwards*, and the reaction of the fulcrum also acts *upwards*, so that the lever presses *downwards* on the fulcrum.

Since $CA > CB$, the effort is less than the weight; so that *the mechanical advantage is always greater than unity.*

And since $AB = AC - BC$, the reaction is equal to the amount by which the weight exceeds the effort, or

$$R' = Q - P.$$

Examples of this class.—A wheelbarrow (Fig. 84), the fulcrum being where the wheel touches the ground; a crowbar, when the lower end rests on the ground.

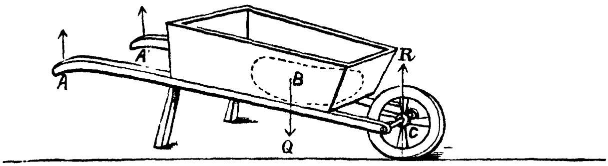


Fig. 84.

Double lever.—A pair of nut-crackers, the forces on the two arms being shown in Fig. 85.

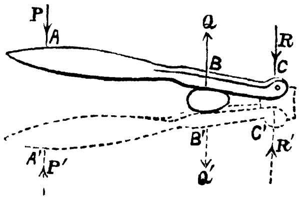


Fig. 85.

An oar **is often described as a lever of the second class. It cannot be strictly said to belong to either class. If the boat were kept at rest and the oar used to scoop the water backwards, it would be a lever of the first class, with the rowlock as fulcrum. When the boat moves forwards instead of the water moving backwards, the relative motion and the relation between the effort applied to the handle and the resistance of the water are the same as before,** ***and can be correctly found by treating the oar as a lever of the first class.********

* This is proved in a note published in the *Philosophical Magazine* for Jan., 1887.

92. A lever of the third class (Fig. 86) is one in which the effort is applied between the fulcrum and the resistance, the points of application, therefore, occurring in the order C, A, B.

Hence CA and AB are in one direction, and BC is in the opposite direction.

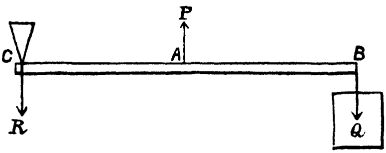


Fig. 86.

Therefore Q, R both act in one direction, and the effort P acts in the opposite direction.

Hence the effort must be applied *upwards*, and the reaction of the fulcrum acts *downwards*; so that the lever presses *upwards* on the fulcrum.

Since $CB > CA$, the effort is greater than the weight; so that *the mechanical advantage is always less than unity.*

And since $AB = CB - CA$, the reaction R is equal and opposite to the excess of the effort over the weight, or

$$R = P - Q.$$

Examples of this class.—The treadle of a turning-lathe or scissors-grinding machine; a door-knocker.

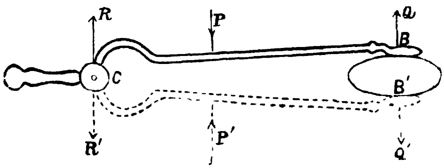


Fig. 87.

Double lever.—A pair of tongs (Fig. 87).

Levers of the third class are rarely used, for, since the mechanical advantage is less than unity, a greater effort is necessary to overcome the resistance than if no lever were used at all. This is sometimes expressed by saying that there is *mechanical disadvantage*. They are therefore useless for raising heavy weights. The use of a pair of tongs is to enable the effort to be applied at a more convenient position.

93. OBSERVATION.—In the solution of problems relating to levers, the student is recommended not to make use of the distinction between the three classes of lever, but to work each case out independently, either by taking moments or by writing down the conditions of equilibrium of the three forces acting on the lever.

Example.—A slab of stone weighing 1 ton, whose weight acts at its centre, is to be tilted up by a crowbar 3 ft. long resting against a log of wood in front of it. To find where the log must be placed in order that a force of 1 cwt. may suffice to raise the slab.

Let ACB be the crowbar, BGD the slab resting on the ground at D, G its centre of gravity. Let P, Q, R be the forces acting on the crowbar at A, B, C.

Then the weight of the stone, 1 ton ($= W$, say), acts at G, and is lifted about D by the force Q acting at B.

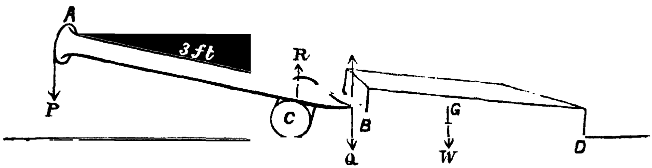


Fig. 88.

Taking moments about D,

$$Q \times BD = W \times GD.$$

But G is the middle point of BD;

$$\therefore \quad Q = W \times \frac{GD}{BD} = 1 \text{ ton} \times \tfrac{1}{2} = \tfrac{1}{2} \text{ ton} = 10 \text{ cwt}.$$

Taking moments about C for the crowbar,

$$P \times AC = Q \times CB.$$

Or, since $P = 1$ cwt., $AC = \dfrac{Q}{P} CB = \dfrac{10}{1} CB = 10CB$;

$$\therefore \quad AB = AC + CB = 10CB + CB = 11CB;$$

$$\therefore \quad CB = \tfrac{1}{11}AB = \tfrac{1}{11} \text{ of } 3 \text{ ft.} = 3\tfrac{3}{11} \text{ ins.}$$

Hence the crowbar must rest on the log at a point $3\frac{3}{11}$ ins. from its extremity.

94. The wheel and axle are two cylindrical rollers joined together with a common axis terminating in two pivots about which they can turn freely. The larger roller is called the **wheel,** and the smaller the **axle.** Both the wheel and the axle have ropes coiled round them in opposite directions. The rope on the axle supports the weight, and the effort is applied by pulling the rope

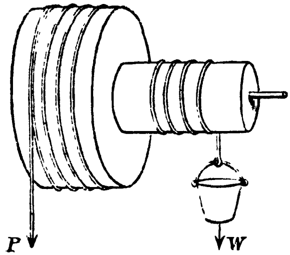


Fig. 89.

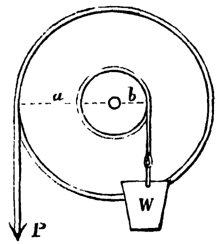


Fig. 90.

attached to the wheel. As the rope round the wheel unwinds, that round the axle winds up and raises the weight. Fig. 90 shows an end view of the arrangement.

Mechanical advantage of the wheel and axle.—The condition of equilibrium is the same as if the strings were really in a vertical plane perpendicular to the common axis as they appear in Fig. 90, and therefore the moments of the effort P and weight W about the axis are equal and opposite. Here, if a denotes the radius of the wheel, and b that of the axle, then a, b are the arms on which P and W act, and therefore

$$Pa = Wb;$$

i.e., effort × rad. of wheel = weight × rad. of axle.

$$\therefore \textbf{ mechanical advantage} = \frac{W}{P} = \frac{a}{b} = \frac{\textbf{rad. of wheel}}{\textbf{rad. of axle}} \quad \ldots\ldots\ldots (2).$$

By making the wheel large and the axle small, the mechanical advantage may be made as great as is desired.

95. The wheel and axle is thus really equivalent to a lever whose arms are the radii of the wheel and the axle. But the lever can only be used for raising weights through short distances; the wheel and axle will lift them to any desired height.

Instead of the rope being coiled round the wheel, an endless rope may be used, passing round a groove cut in the rim of the wheel, as in a common roller-blind, provided proper precautions are taken to prevent the rope from slipping round in the groove.

96. The windlass (Fig. 91) is a modification of the wheel and axle, the only difference being that the effort is applied by turning a handle AH at the end of an arm CA.

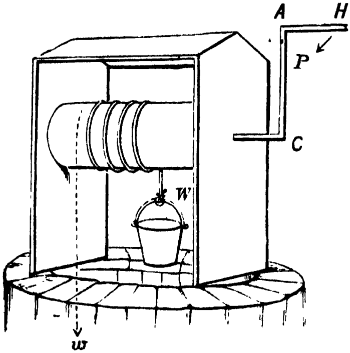


Fig. 91.

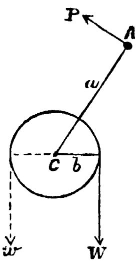


Fig. 92.

It is commonly used for raising buckets of water from a well, or earth from a shaft. An improved form has two buckets so arranged that the empty one goes down as the full one comes up.

Mechanical advantage.—If a is the length of the arm CA, the equation of moments gives, as before,

$$Pa = Wb;$$

and $\therefore$ mech. advantage $= \dfrac{a}{b} = \dfrac{\text{length of arm}}{\text{radius of axle}}$,

the length of the arm thus taking the place of the radius of the wheel.

If there are two buckets, and the total ascending and descending weights are W and w, we shall have, by taking moments,

$$Pa = Wb - wb = (W - w)b.$$

Example. — The axle of a windlass is 8 ins. in diameter, and carries two buckets of equal weight on opposite sides. To find the force which must be applied to a handle, whose arm is 2 ft., to raise 3 gallons of water, a gallon weighing 10 lbs.

Let P be the force, w the weight of each bucket. Then, since the radius of the axle is 4 ins., the arm of the handle 24 ins., and the total weights on the two sides w and $w + 30$ lbs., we have, by moments,

$$P \times 24 = (w + 30) \times 4 - w \times 4 = 30 \times 4;$$

whence $$P = 5 \text{ lbs.}$$

[Notice that the weights of the two buckets balance each other.]

97. The capstan (Fig. 93) used on board ship is exactly similar in principle, but the barrel turns on a

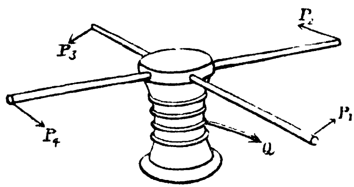


Fig. 93

vertical axis and is worked by one or more men walking round and pushing a number of horizontal projecting arms (called handspikes). Here the moment of the pull of the rope is equal to the *sum* of the moments of the forces exerted by the men.

Example. — The barrel of a capstan is 3 ft. in diameter, and is worked by four men exerting forces of 45, 52, 63, and 64 lbs. on arms each $7\frac{1}{2}$ ft. long. The rope passing round the barrel is fastened to a pier. To find the force drawing the ship towards the pier.

Let Q lbs. be the required force exerted by the rope. Then, since the radius of the barrel is $1\frac{1}{2}$ ft., we have, by taking moments,

$$Q \times 1\tfrac{1}{2} = (45 + 52 + 63 + 64) \times 7\tfrac{1}{2};$$

$$\therefore \quad Q = 224 \times 5 \text{ lbs.} = 1120 \text{ lbs.} = \tfrac{1}{2} \text{ ton.}$$

98. **Principle of Work for any machine.**—Although a small effort may be made to overcome a very large resistance with a machine, the Principle of Work, or Principle of Conservation of Energy, holds good in every case, and asserts that the work done by the effort is always equal to the work done by the machine against the weight or resistance.

Hence no work is gained or lost by the use of a frictionless machine.

For instance, if, in any machine, a force of 1 lb. supports a weight of 10 lbs., the former force will have to move its point of application through 10 ft. to raise the weight through 1 ft.

This is sometimes expressed by saying that "what is gained in power is lost in speed." In more accurate language, mechanical advantage is always obtained at the expense of a proportionate disadvantage in diminished speed.

Conversely, where increased speed is obtained by means of a machine, this is only attained at the expense of mechanical disadvantage.

Example.—The arms of a lever are 3 ft. and 1 ft. To find the force on the longer arm and the work done in raising a weight of 12 lbs. through 1 in., and to verify the Principle of Work.

Let P be the required force. Then, by taking moments about the fulcrum,

$$P \times 3 = 12 \times 1, \quad \text{whence} \quad P = 4 \text{ lbs.}$$

Let the lever be turned about the fulcrum. Then the points furthest from the fulcrum will move over the greater distances; and, by drawing a figure with the lever in two positions, it is easy to see, or to prove, by Euclid VI. 6, that the distances moved by different points are proportional to their distances from the fulcrum. Thus, if the end of the shorter arm moves 1 in., that of the longer arm will move 3 in.

Now work required to lift 12 lbs. through 1 in.

$$= 12 \times \tfrac{1}{12} = 1 \text{ ft.-lb.}$$

Work done by P, or 4 lbs., in moving its point of application through 3 ins. $= 4 \times \tfrac{3}{12} = 1$ ft.-lb.

$\therefore$ work done by P = work required to raise weight.

Therefore the Principle of Work is true in this case.

99. **Principle of Work for the wheel and axle.**—Let a, b be the radii of the wheel and axle, and let them be rotated through one complete turn. Then a length of rope equal to the circumference of the wheel, or $2\pi a$, uncoils from off the wheel, and a length equal to the circumference of the axle, or $2\pi b$, coils round the axle.

Hence, distance *fallen* by P (the effort)

$$= \text{circumference of wheel} = 2\pi a,$$

and distance *risen* by W (the weight)

$$= \text{circumference of axle} = 2\pi b;$$

$$\therefore \quad \text{work done by } P = P \times 2\pi a,$$

and work done against $W = W \times 2\pi b$.

(i.) *If we assume the equation of moments*

$$P \times a = W \times b,$$

then $$P \times 2\pi a = W \times 2\pi b,$$

or $$\text{work done by } P = \text{work done against } W,$$

verifying the truth of the Principle of Work for the wheel and axle.

(ii.) Conversely, *if we assume the Principle of Work to be true*, then $$P \times a = W \times b,$$

verifying the relation between the effort and resistance, which is otherwise obtainable from the equation of moments.

100. **To find the mechanical advantage of any machine from the Principle of Work.**

We shall now show that the mechanical advantage or the condition of equilibrium of a machine working without friction can very easily be found by means of the Principle of Work when they *cannot* be easily found by other methods.

For let the machine be set in motion. Then it is only necessary to compare the distances through which the points of application of the effort and resistance move; their ratio is the required mechanical advantage.

For if P, Q denote the effort and resistance, x, y the distances moved by their points of application, then the

relation work done by P = work done against Q

gives $P \times x = Q \times y$.

$\therefore$ mechanical advantage, $\dfrac{Q}{P} = \dfrac{x}{y}$

$$= \frac{\text{distance moved by point of application of } P}{\text{distance moved by point of application of } Q}.$$

Examples.—If, by moving a handle through 1 ft., a weight of 1 cwt. is raised through 1 in., to find the force that must be applied to the handle.

The mechanical advantage is the ratio of 1 ft. to 1 in., and is therefore 12.

Hence force required to raise 112 lbs. = 112/12 = $9\frac{1}{3}$ lbs. weight.

101. **Toothed or cog- wheels.**—By the use of toothed wheels, the mechanical advantage of a windlass or other similar machine can be increased to any desired extent. By counting the number of teeth or cogs on two wheels which work into one another, we may find the number of turns and fractions of a turn made by one wheel for each turn of the other, and in this way determine the relation between the effort and weight.

Examples.—(1) The handle of a windlass is 1 ft. long, and is connected to a cog-wheel with 6 teeth, which works another cog-wheel with 84 teeth, connected to an axle 9 ins. in diameter on which is coiled the rope supporting the weight. To find the force which must be applied to the handle to lift 5 cwt.

Since the cog-wheels contain 6 and 84 teeth, respectively,

$\therefore$ one turn of the axle corresponds to $\frac{84}{6}$ or 14 turns of the handle.

In one turn of the handle the extremity describes a circle of circumference 2π ft., and in one turn of the axle a length $\frac{3}{4}\pi$ ft. of rope coils up, drawing up the weight.

Therefore, by the Principle of Work, if P is the required force in lbs. weight, $P \times 14 \times \pi \times 2 = 560 \times \pi \times \frac{3}{4}$.

$$\therefore\ P = \frac{560 \times 3}{14 \times 2 \times 4} = 15 \text{ lbs. weight.}$$

(2) The driving wheel of a bicycle is 90 ins. in circumference, the cranks of the pedals are $4\frac{1}{2}$ ins. long, and the driving wheel makes 20 turns for every 9 turns of the crank axle. If the force resisting the motion of the machine is $\frac{1}{2}$ lb., to find the average force which

the rider exerts on the pedals, supposing him to press on them vertically downwards.

In each turn of the driving wheel the machine moves forward 90 ins.

But the turn of the cranks produces $\frac{20}{9}$ turns of the driving wheel.

Therefore in one turn of the *cranks* the machine moves forward 200 ins. $= \frac{50}{3}$ feet, and the work done against the resistance is $\frac{50}{3} \times \frac{1}{2}$ or $\frac{25}{3}$ ft-lbs. weight.

But in one turn of the cranks each of the two pedals is lowered in turn through a vertical distance = twice the length of the crank $= 9$ ins. $= \frac{3}{4}$ ft.

Hence, if P lbs. denote the average force which the rider exerts on the pedals, the Principle of Work gives

$$P \times 2 \times \tfrac{3}{4} = \tfrac{25}{3}.$$

$$\therefore \quad P = \tfrac{50}{9} = 5 \cdot \dot{5} \text{ lbs. weight.}$$

NOTE.—The student is at liberty to apply the Principle of Work or Principle of Conservation of Energy to any problem whatever in Mechanics, provided its use is not precluded by the conditions of the question (as, for example, where it is required to *verify* the principle when its truth must not be *assumed*).

SUMMARY OF RESULTS.

Mechanical advantage of any machine is defined as fraction

$$\frac{\text{resistance to be overcome}}{\text{effort applied to move machine}} = \frac{Q}{P} \text{ or } \frac{W}{P}. \qquad (\S\, 85.)$$

Let this be denoted by M, while Q or W and P have their usual meanings.

In any *lever* the equation of moments about fulcrum gives $M = \dfrac{Q}{P} = \dfrac{\text{distance of } P \text{ from fulcrum}}{\text{distance of } Q \text{ from fulcrum}} \ldots (1).$ (§ 87.)

For a *straight lever* ABC, whose fulcrum is C, the conditions of equilibrium are

$$\frac{P}{BC} = \frac{Q}{CA} = \frac{R}{AB}, \qquad (\S\, 89.)$$

where R is the reaction of the fulcrum.

In a lever of the

first	class, *fulcrum* is in middle,		$M >=$ or <1,	(§ 90.)
second	„ *resistance* „	„	M always >1,	(§ 91.)
third	„ *effort* „	„	M always <1.	(§ 92.)

In a *wheel and axle* or *windlass*, $Pa = Qb$, or

$$M = \frac{a}{b} = \frac{\text{rad. of wheel}}{\text{rad. of axle}} \text{ or } \frac{\text{arm of handle}}{\text{rad. of axle}} \quad \ldots\ldots \quad (2).$$

(§§ 94, 96).

The Principle of Work for any machine gives

work of effort = work against resistance ;

whence, if x, y are distances traversed by P, Q,

$$Px = Qy, \quad M = \frac{x}{y}. \qquad (\S\ 100.)$$

EXAMPLES VII.

1. The arms of a lever are 4 ins. and 11 ins. in length, and a weight of $41\frac{1}{4}$ lbs. is attached to the shorter arm. Find the power.

2. A lever, 6 ft. long, having the fulcrum at one end and the applied force at the other end, is used to sustain a weight of 1 cwt. at a point 2 ft. from the fulcrum. Find the direction and magnitude of the pressure on the fulcrum.

3. Two weights P and Q balance on a weightless lever, the fulcrum being $1\frac{1}{4}$ ins. from the middle point of the lever. If each weight is increased by 1 lb., the fulcrum must be moved $\frac{1}{4}$ in. in order that there may be equilibrium. Find the force of pressure on the fulcrum in each case.

4. Explain carefully why a man stands on the bottom rung of a ladder, and holds on to another rung as low down as he can, when another man is lifting the ladder.

5. Two men carry a load of 1 cwt. suspended from a horizontal pole 12 ft. long, whose weight is 20 lbs., and whose ends rest on their shoulders. Find the point at which the load must be suspended in order that one of the men may bear 94 lbs. of the whole weight.

6. The drum of a windlass is 4 ins. in diameter, and the power is applied to the handle 20 ins. from the axis. Find the force necessary to sustain the weight of 100 lbs., and the work done in turning the handle ten times.

7. Why cannot a man, sitting in a basket, lift himself and the basket off the ground by pulling at the handles of the basket?

8. What power will balance a weight of 6 cwt. by means of a wheel and axle whose circumferences are, respectively, 5 ft. 4 ins. and 9 ins.?

9. A wheel and axle is used to raise a bucket weighing 30 lbs. from a well. The radius of the wheel is 20 ins., and while it makes 7 revolutions the bucket rises 11 ft. What is the smallest force that will raise the bucket?

10. Two forces P and Q balance on a lever acting on the same side of the fulcrum. If P be increased by 1 lb., equilibrium may be maintained by moving P's point of application 2 ins.; or if Q be increased by 1 lb., by moving Q's point of application $\frac{3}{4}$ in.; or if both P and Q be increased by 1 lb., by moving both points of application $1\frac{1}{3}$ ins. Find P and Q.

11. In any lever, find the mechanical advantage when the effort is applied at (i.) the point of application of the resistance, (ii.) the fulcrum. Applying this to the case of an oar in a boat, state what happens when the oarsman pulls hold of it *at* the rowlock, and hence show that the oar acts as a lever whose fulcrum is at the rowlock.

12. AB is a weightless lever acted on at A and B by two equal forces P and Q, whose directions contain an angle $60°$, P acting at right angles to AB. Find where the fulcrum must be situated so that P and Q may be in equilibrium, and find the force of pressure they exert on the fulcrum.

13. AB is a weightless rod turning freely round its middle point C; D is a point vertically under C, such that CD equals half the length of the rod; the points B and D are connected by a thread of the same length as CD or CB. If a weight W is hung from A, what is the tension of the thread BD, and what is the magnitude and direction of the pressure on the point C?

14. Two weights of 4 lbs. and 8 lbs. balance when suspended from the ends of a straight lever with the fulcrum 1 ft. from the larger weight. When P lbs. are added to each weight, the fulcrum has to be shifted a distance of 2 ins. Find the value of P and the length of the lever.

15. Draw to scale a wheel and axle by which a man, sitting in a loop at the end of a rope wound round the axle, can haul himself up by pulling at a rope round the wheel with a force only one-fifth of his weight. What weight is sustained by the pivots?

16. State the principle of the Conservation of Energy, and show how it holds good in the case of a lever of the third kind.

EXAMINATION PAPER III.

1. What do you mean by the *moment* of a force about a point?

2. Show that the algebraical sum of the moments of two forces (whose lines of action intersect) about any point in the plane containing the forces is equal to the moment of their resultant.

Deduce the rule for compounding two like parallel forces.

3. Find the resultant of two parallel forces acting in opposite directions.

4. *ABC* is a triangle having a right angle at *C*; *BC* is 12 ft., and *AC* is 20 ft.; *P* is a point in the hypotenuse *AB* such that *AP* is one-fourth of *AB*; a force of 50 lbs. acts from *C* to *B*, and one of 100 lbs. from *C* to *A*. (*a*) Find the moments of the forces with respect to *P*. (*b*) Find the sum (*i.e.*, the algebraical sum) of the two moments. (*c*) If the point *P* were fixed, in what direction would the forces make the triangle revolve?

5. State the conditions of equilibrium of three parallel forces acting upon a rigid body.

6. A straight line *AB* represents a rod, 10 ft. long, supported horizontally on two points, one under each end; *C* is a point in *AB*, 3 ft. from *A*. What thrust is produced on the points *A* and *B* by a weight of 30 lbs. hung at *C*? What additional thrust is exerted on the points of support if the rod is uniform and weighs 20 lbs.?

7. Classify the different kinds of levers, pointing out those in which there is a mechanical advantage.

8. Explain how the mechanical advantage of an oar will be altered by altering the position of the point in contact with the rowlock.

9. "What is gained in power is lost in speed."—Explain clearly what is meant by this statement, and show that it is true in the case of the wheel and axle.

10. In a wheel and axle the radius of the wheel is 2 ft. and that of the axle is 2 ins. What power will balance a weight of 97 lbs. if the ropes to which the power and weight are applied are each $\frac{1}{2}$ in. in diameter?

CHAPTER VIII.

MACHINES—THE PULLEY AND SYSTEMS OF PULLEYS.

102. The pulley, or pully,* is a wheel with a *groove* cut round its rim so that it can carry a string or rope or chain passing round it.† Pulleys turn on *pivots,* which are attached to a framework called a *block* or *sheave,* and this block is either **fixed,** or is attached to a string and is then **moveable.**

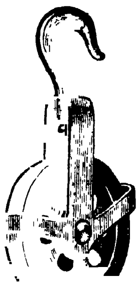


Fig. 94.

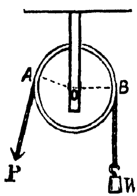


Fig. 95.

103. In the fixed pulley the weight is attached to one end of the string passing round the groove, and the effort is applied by pulling the other end. Since the wheel is only supported at its centre by the pivots, the moments of the effort and weight about the centre O are equal and opposite; that is,

* The word may either be spelt *pulley,* plural *pulleys,* or *pully,* plural *pullies.*

† For the sake of uniformity we shall speak of a *string.* The weight of this string will be assumed too small to require to be taken into account.

$$P \times OA = Q \times OB,$$

or $$P \times \text{radius of pulley} = Q \times \text{radius of pulley}.$$

$$\therefore \quad \textbf{effort } P = \textbf{weight } Q,$$

$$\therefore \quad \textbf{mechanical advantage } \frac{Q}{P} = 1 \quad \ldots\ldots\ldots \quad (1).$$

Thus exactly the same force must be applied to lift a given weight as if the weight were lifted without the pulley. The only difference is that the force can be applied in a different direction; hence the usefulness of the fixed pulley lies in its convenience only.

Thus, in raising building materials to the top of a house, it is far easier for a man at the bottom to pull down a rope passing over a pulley at the top than it would be for a man at the top to hoist them up with a rope.

104. In the single moveable pulley, the weight is attached to the block, and the effort is applied to one end of the string which passes round the pulley, the other end being fixed up.

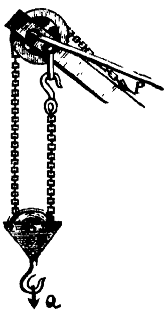


Fig. 96.

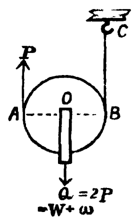


Fig. 97.

In this arrangement, if the strings are parallel, the weight is double the effort and the mechanical advantage is 2.

For let P be the effort, Q the total weight to be raised (*including the weight of the pulley itself*). By taking moments about the point B where the fixed-up end leaves the pulley, we have

$$Q \times OB = P \times AB,$$

$$Q \times \text{radius of pulley} = P \times \text{diameter of pulley}.$$

$$\therefore \quad Q = 2P,$$

and mechanical advantage $Q \div P = 2$ .... (2).

This also follows from the fact that the weight is supported by the two pulls in the two parts of the string at A, C, each of which pulls is equal to P.

The single moveable pulley is much used on cranes (Fig. 96) for the purpose of doubling the mechanical advantage.

105. The single-string system of pulleys.* — A greater mechanical advantage may be obtained with an arrangement containing a number of pulleys. Several such "**systems of pulleys**" are generally described, but the most practically useful system is that figured in Figs. 98, 99, in which a number of pulleys are arranged in two blocks, one fixed and the other attached to the weight or resistance. The same string passes round all the pulleys; it passes alternately round a fixed and round a moveable pulley, and is finally attached to one or other of the two blocks.

In practice the pulleys are arranged as in Fig. 98, but it is generally easier to draw the diagram for an arrangement like Fig. 99, in which all the pulleys are in the same plane.

Mechanical advantage.—If the string be pulled with a force P, the pull in every part of the string is P. Hence, if n be the number of parts of the string supporting the lower block, Q the lower weight to be raised (including that of the lower block and its pulleys), the n forces P acting upwards are in equilibrium with Q acting

---

* The single-string system is often called the *second system* of pulleys, and the separate-string system, described in § 106, is then called the *first system*. These names should be remembered in deference to some examiners, but it is more convenient to describe them in the present order on account of the similarity between the so-called first and third systems.

downwards; hence, supposing the parts of the string vertical, the conditions of equilibrium give

$$Q = nP.$$

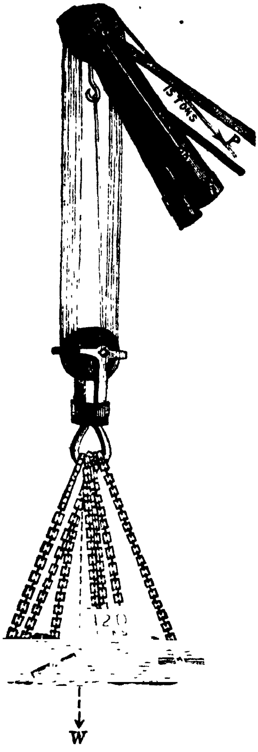


Fig. 98.

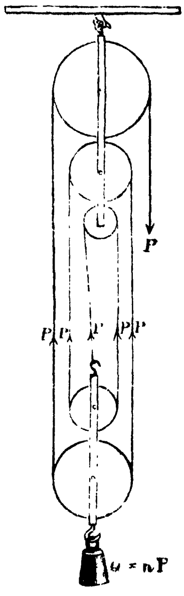


Fig. 99.

$\therefore$ **mechanical advantage** $\dfrac{Q}{P} = n$ .... (3).

In order that the effort may be applied *downwards*, the free end of the string must hang from a fixed pulley, and

this is almost invariably done for convenience in working the system. In such cases the number n is also the total number of pulleys in the two blocks.

Thus Fig. 98 represents a system with altogether 8 pulleys, in which the mechanical advantage is therefore 8. In Fig. 99, there are altogether 5 pulleys, and the mechanical advantage is 5.

[It will be found that the mechanical advantage is measured by an even or odd number, according to whether the end of the string is finally attached to the fixed or to the moveable block.]

Example.—To find the least number of pulleys in a moveable block weighing 10 lbs., in order that a weight of 120 lbs. may be lifted by a downward force not exceeding 28 lbs., and to find this force.

Let n be the total number of portions of the string supporting the lower block, P the required effort. Then the pulls P in the strings have to support both the attached weight of 120 lbs. and the block weighing 10 lbs.; therefore

$$nP = 120 + 10 = 130 \text{ lbs.}$$

But P is not more than 28 lbs.; therefore nP, or 130, is not more than $28n$, or n is a whole number not less than $\frac{130}{28}$, that is, $4\frac{9}{14}$. Therefore $n = 5$.

Hence five parts of the string must support the lower block. Therefore that block must contain two pulleys, and must have the end of the string attached to it as well (Fig. 99). Also, putting $n = 5$, we have $5P = 130$ lbs.; $\therefore$ required force $P = 26$ lbs.

106. **The separate-string system of pulleys*** consists of a number of single moveable pulleys like that described in § 104, so arranged that the string hanging from one pulley passes round the pulley next below, the other ends of the strings being attached to a fixed beam or other support (such as the mast of a ship), considerably above the highest points to which weights have to be raised (Fig. 100).

The mechanical advantage may be found as follows:—

In the single moveable pulley a force P applied to the string supports a force $2P$ applied to the block.

Now suppose the moveable pulley, instead of being attached to the weight, supports a string passing over a second moveable pulley. Then the mechanical advantage

* The so-called *first system.*

gained by the first pulley is evidently doubled by the second, the pull $2P$ in the second string supporting a weight $4P$ attached to the second pulley.

Next suppose the second pulley supports a string passing over a third pulley. This again doubles the mechanical advantage, and the system will now support a weight $8P$.

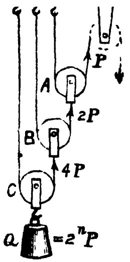


Fig. 100.

In this way each additional pulley doubles the mechanical advantage of the system. By using 1, 2, 3 pulleys, we get mechanical advantages represented by the numbers 2, 4, 8.

Generally, let there be n pulleys, and let Q denote the weight attached to the last pulley. Then, if we leave out of account the weights of the pulleys themselves, we have

$$Q = 2^n P.$$

Therefore also $$P = \frac{Q}{2^n}$$

and **mechanical advantage** $= 2^n$ .... (4).

Examples.—(1) If there are ($n =$) 4 moveable pulleys, a force of ($P =$) 10 lbs. will support a weight

$$(Q =)\ 2^4 P = 2^4 \times 10 \text{ lbs.} = 160 \text{ lbs.}$$

(2) If there are ($n =$) 3 moveable pulleys, the force required to support a weight ($Q =$) 64 lbs. is

$$(P =)\ Q \div 2^n = 64 \div 2^3 = 64 \div 8 = 8 \text{ lbs.}$$

107. *If the weights of the pulleys have to be taken into account*, it is most convenient to calculate the relation between the effort and weight, as in the following examples:—

Examples.—(1) What weight can be supported by a force of 10 lbs. in a system of 3 moveable pulleys whose weights, beginning with the highest, are 1, 2, 3 lbs., respectively?

Consider, first, the equilibrium of the highest pulley (A, Fig. 100). The forces on it are the two equal pulls of 10 lbs. in the two parts of the string round it, acting upwards, and the weight of the pulley (1 lb.) and the tension of the string next below, acting downwards. Hence twice the pull of the first string is equal to the weight of the top pulley *plus* the pull of the second string. Applying similar reasoning to the other pulleys in succession, we may arrange the process thus:

Pull of string round first (highest) pulley A	= 10 lbs.
Multiply by	2
Total force supported at A, due to weight of pulley and tension of second string	= 20 ,,
Subtract weight of first pulley A	1 ,,
Therefore pull of string round second pulley B	= 19 ,,
Multiply by	2
Total force supported at B	= 38
Subtract weight of second pulley B	2
Pull of string round third pulley C	= 36
Multiply by	2
Total force supported at C	= 72
Subtract weight of third pulley C	3
Required weight supported by system	= 69 lbs.

(2) What force is required to support a weight of 13 lbs. in a system of four moveable pulleys whose weights, commencing with the highest, are 3, 5, 7, 9 lbs., respectively?

Here we are given the weight, and have to find the effort. Hence the process is the reverse of that of Ex. (1). We must begin with the lowest pulley. Adding its weight (9 lbs.) to the attached weight (13 lbs.), we have the total weight supported by the two parts of the string round the lowest pulley, and, since the pulls in these two parts

are equal, each is half the total weight. Applying similar reasoning to each pulley, we may arrange the process as follows, the steps being those of Ex. (1) worked backwards:—

Weight	= 13 lbs.
Add weight of lowest pulley	9 ,,
Total force supported by strings round lowest pulley	= 22 ,,
Divide by 2.	
∴ pull of string round lowest pulley	= 11 ,,
Add weight of next pulley	7 ,,
	2) 18 ,,
∴ pull of string round next pulley	= 9 ,,
Add weight of third pulley from bottom	5 ,,
	2) 14 ,,
∴ pull of string round third pulley from bottom	= 7 ,,
Add weight of top pulley	3 ,,
	2) 10 ,,
∴ pull of string round the top pulley	= 5 lbs.
That is, the required force	= 5 lbs.

*108. **The inverted separate-string system of pulleys,*** in which the strings are all attached to the weight, is merely the system last described (*i.e.*, the "first") turned upside down. In the separate-string system, the strings were all attached to the supporting beam, and the lowest pulley supported the weight. In the present system, the strings are all attached to the weight, or rather to a rod carrying the weight, and the uppermost pulley fixed to some support. If Fig. 100 be turned upside down (the fixed pulley being omitted), it will present a similar appearance to Fig. 101.

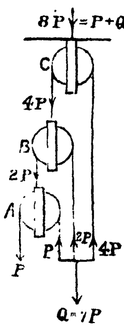


Fig 101.

* The so-called *third system*. This system is practically useless (see § 110).

The mechanical advantage (when the weights of the individual pulleys are neglected) is easily deduced from this property. Let R be the pull which the system exerts on its support, Q the weight, P the effort, and let there be n pulleys. By inverting the system or otherwise, we see that the total forces supported by the several pulleys, commencing with the lowest, are $2P$, $4P$, $8P$, &c.; thus R in this present system corresponds to the weight in the last system, and therefore $R = 2^n P$.

Now consider the equilibrium of the whole system, consisting of the weight and the pulleys. The forces acting on it are Q and P pulling downwards *and a reaction equal and opposite to R holding the system up.* Hence, since these forces keep the system in equilibrium, therefore

$$R = P + Q.$$

$$\therefore \quad Q = R - P = 2^n P - P = \mathbf{(2^n - 1)P},$$

and $$\textbf{mechanical advantage} = Q \div P = \mathbf{2^n - 1} \quad \ldots(\mathbf{5}).$$

OBSERVATION.—The pulls of the strings, beginning with the lowest, are P, $2P$, $4P$, $8P$, ... $2^{n-1}P$, and, since these support the weight Q, we have $Q = P(1+2+4+8+\ldots+2^{n-1})$. Now $1+1=2$; therefore $1+1+2=2+2=4$; therefore $1+1+2+4=4+4=8$ and so on; therefore $1+1+2+\ldots+2^{n-1} = 2^{n-1}+2^{n-1} = 2\times 2^{n-1} = 2^n$ or $1+2+4+\ldots+2^{n-1} = 2^n - 1$, giving $Q = (2^n - 1)P$, as above.

Example.—To find the number of weightless pulleys, having given that a force of 5 lbs. supports a weight of 75 lbs., and all the strings being attached to the weight.

Here the pull on the beam supporting the upper pulley

$$= 75 \text{ lbs.} + 5 \text{ lbs.} = 80 \text{ lbs.}$$

Now $80 = 5 \times (2 \times 2 \times 2 \times 2)$, and, since the total pull is doubled by each pulley, the number of pulleys must be 4.

109. *If the weights of the pulleys are taken into account*, we proceed as follows:—

Examples.—(1) To find the weight which can be lifted by a force of 10 lbs. in a system of three pulleys, the weights of the lowest and next pulleys being 3 and 5 lbs.,* and the strings all being attached to the weight.

* Since the upper pulley is *fixed*, its weight does not enter into the relation between P and W.

Here the pulls in the two parts of the lowest string are each 10 lbs. The next string above has to support these pulls, and also to support the weight of the lowest pulley, viz., 3 lbs., and so on; hence the process stands thus:

Pull of lowest string	= 10 lbs.	= 10 lbs.
Multiply by	2	
	20 "	
Add weight of lowest pulley	3 "	
∴ pull of second lowest string	= 23 "	= 23 "
Multiply by	2	
	46 "	
Add weight of second pulley	5 "	
∴ pull in last string	= 51 "	= 51 "

Required weight = sum of pulls of strings = 84 lbs.

(2) To find the force required to support a weight of 112 lbs. in the system of Ex. (1).

Let P be the force. Proceed, as in Ex. (1), thus:

Pull of lowest string	$= P$	$= P$
	2	
	$2P$	
	3	
Pull in second lowest string	$= 2P+3$	$= 2P+3$
	2	
	$4P+6$	
	5	
Pull in last string	$4P+11$	$= 4P+11$

$\therefore$ weight supported $= 7P+14$

$\therefore \quad 7P+14 = 112.$

$\therefore$ required force $P = 14$ lbs.

110. **Uses of the various systems of pulleys.**—The single-string system, with the pulleys arranged in two blocks, as in Fig. 98, is by far the most important system for practical use, because with it weights may be raised till the two blocks come together, *i.e.*, nearly up to

the level of the supporting beam or crane, or lowered till all the string is paid out.

The separate-string system gives a greater mechanical advantage with the same number of pulleys, for each additional pulley doubles the mechanical advantage instead of merely increasing it by unity. But it requires the strings to be attached to a point considerably above the highest point to which weights have to be raised, for when the top pulley is drawn up as high as it will go the weight will still be a considerable distance below the support. By attaching the strings (or rather ropes) high up on the mast of a ship, the system may be used, and *is* actually so used, for raising cargo from the hold up to the deck.

The inverted separate-string system, or "third system," is of no practical use whatever, for it is found that the strings are almost certain to get hopelessly entangled, even in the few working models that are constructed for the lecture-room.

[This system seems to have been originally introduced into text-books and examination papers on account of its being a convenient subject for problems illustrating the summation of a geometrical progression.

Generally speaking, calculations taking account of the weights of the pulleys are of theoretical rather than practical interest, because the weights of the strings and the friction (which are *not* taken into account) are actually quite as important as the weights of the pulleys.]

111. Man raising himself with a system of pulleys.—When a man, sitting in a loop or seat suspended by any arrangement of pulleys, pulls *himself* up, the rope which he pulls will support part of his weight, and only the remaining part of his weight will have to be supported by the system.

Examples.—(1) Consider a man of weight W raising himself by pulling a rope passing over a *fixed* pulley with a force P. The man's weight is really supported by the pulls P in *two* parts of the rope—the part where he pulls and the part supporting the loop. Hence the condition of equilibrium gives $W = 2P$;

$$\therefore \quad P = \tfrac{1}{2}W;$$

or the man pulls with a force of *half* his weight.

(2) If the man pulls a rope which passes over a fixed pulley and under a movable pulley supporting the loop, there are three portions

of the string supporting his weight, inclusive of the one that he is pulling. Hence the relation between the pulling force P and the weight is $W = 3P$, or $P = \frac{1}{3}W$; so that the man pulls with a force of *one-third* of his weight.

Generally, suppose that a man is pulling himself up by applying a force P to a rope passing round a system of pulleys or a wheel and axle, in which P will support a weight nP. Then, since the rope he pulls also helps lift him up with a force P, the total weight supported must be $nP+P$ or $(n+1)P$. Hence

$$\textbf{mechanical advantage} = n+1 \quad \ldots\ldots\ldots(\mathbf{6});$$

instead of being n, as it would be for a man standing on the ground and raising a weight.

112. **Applications of the Principle of Work.**—We shall now apply the Principle of Work to find the relations between the effort and weight in the various systems of pulleys, and shall, at the same time, take separate account of the weights of the pulleys.

In order to make a distinction between formulæ which do not and those which do take account of the weights of the separate pulleys, the weight raised has been denoted by Q in the former, and will be denoted by W in the latter, formulæ.

113. **The single moveable pulley.**—Let the pulley (weight w) and its attached weight W be raised through a height h [say, for argument, 1 in.]. Then the portions of string on the two sides of the pulley will each have to be shortened by h; hence the end of the string at which P is applied must move through a distance $2h$ [2 ins.]. Since

sum of works done against W, w = work done by P;

$$\therefore \quad Wh+wh = P\times 2h,$$

or
$$\boldsymbol{W+w = 2P} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(\mathbf{7}),$$

the required relation between W and P. This agrees with (**2**) on writing $Q = W+w$.

114. **The single-string system.** — Let there be n strings from which the lower block of pulleys hangs. If the effort P lifts a weight W, hanging from a moveable block of weight w, through a height h, each of the n portions of string will have shortened by an amount h, and so P will have to pull the end of the string down through a distance nh. By the Principle of Work, therefore,

$$Wh + wh = nPh.$$

$$\therefore \quad \mathbf{W + w = nP} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \mathbf{(8)},$$

the required relation between W and P, which agrees with **(3)** on writing $Q = W + w$.

*115. **The separate-string system.**—Let w_1 be the weight of the pulley supporting the weight W; let w_2, w_3, ... be the weights of the pulleys next above. If W is drawn up through a height h, then the lowest pulley w_1 will rise through h, the next above w_2 will rise through $2h$, the next w_3 through $4h$, and so on; and if there are n moveable pulleys the force P will move its point of application through a distance $2^n h$.

Therefore, by the Principle of Work,

$$P \times 2^n h = Wh + w_1 h + 2w_2 h + 4w_3 h + \ldots + 2^{n-1} w_n h,$$

or $$2^n P = W + w_1 + 2w_2 + 4w_3 + \ldots + 2^{n-1} w_n;$$

or again $$\mathbf{P = \frac{W}{2^n} + \frac{w_1}{2^n} + \frac{w_2}{2^{n-1}} + \frac{w_3}{2^{n-2}} + \ldots + \frac{w_n}{2}} \quad \ldots\ldots \quad \mathbf{(9)}.$$

*116. **The inverted separate-string system.**—In the system of § 115, when the weight rises through a height h, the various pulleys rise through distances h, $2h$, $4h$, ... $2^{n-1}h$, and the point of application of P through $2^n h$. Hence *their distances from the weight w* increase by amounts 0, $(2-1)h$, $(4-1)h$ .. $(2^{n-1}-1)h$, $(2^n-1)h$, respectively.

Now turn the system upside down; fix the pulley that previously supported the weight, and attach the weight to the beam that was previously fixed. We now get the "third" system, and we see that when the weight rises through h the pulleys, beginning with the top, descend through distances 0, $(2-1)h$, $(4-1)h$ ... $(2^{n-1}-1)h$, and the point of application of P through $(2^n-1)h$. Hence if w_1, w_2, ... are the weights of the pulleys, the Principle of Work gives

$$Wh = w_1 + 0h + w_2 \times h + w_3 \times 3h + w_4 \times 7h + \ldots + w_n \times (2^{n-1}-1)h + P \times (2^n-1)h,$$

or $$\mathbf{W = (2^n - 1) P + w_2 + 3w_3 + 7w_4 + \ldots (2^{n-1}-1) w_n} \quad \ldots \quad \mathbf{(10)}.$$

the required relation connecting P and W.

SUMMARY OF RESULTS.

In *fixed* pulley,

$Q = P$, mech. advantage $M = 1$ ... (1). (§ 103.)

With a *single moveable pulley*,

$Q = 2P$, $M = 2$ (pulley light) ... (2), (§ 104.)

$W + w = 2P$ (pulley heavy) ... (7). (§ 113.)

The "*second*" system is single string system, pulleys in two blocks. With n pulleys and string hanging from upper block,

$Q = nP$, $M = n$ (pulleys light) ... (3), (§ 105.)

$W + w = nP$ (pulleys heavy) ... (8). (§ 114.)

The "*first*" is the separate string system, with strings tied up to fixed support. With n moveable pulleys,

$Q = 2^nP$, $M = 2^n$ (pulleys light) ... (4). (§ 106.)

$W + w_1 + 2w_2 + \ldots + 2^{n-1}w_n = 2^nP$ (pulleys heavy, lowest w_1) ... (9). (§ 115.)

The "*third*" is the previous system turned upside down, strings all attached to weight.

$Q = 2^nP - P = (2^n - 1)P$, $M = 2^n - 1$ (pulleys light) ... (5). (§ 108.)

$W = (2^n - 1)P + w_2 + 3w_3 + \ldots (2^{n-1} - 1)w_n$ (pulleys heavy, lowest w_n) ... (10). (§ 116.)

For man pulling himself up, mechanical advantage is increased by unity .... (6). (§ 111.)

EXAMPLES VIII.

1. Find the ratio of the power to the weight in that system of pulleys in which each moveable pulley hangs by a separate string, the number of moveable pulleys being 4.

   If the weights of the pulleys be taken into account, and are 1, 2, 3, and 4 lbs., respectively, beginning with the highest, find what power will support a weight of 294 lbs.

2. Find the ratio of the power to the weight in that system of pulleys in which each string is attached to the weight, the number of moveable pulleys being 4.

   If the weights of the moveable pulleys be taken into account, and are 1, 2, 3, and 4 lbs., respectively, beginning with the lowest, find what power will support a weight of $4\frac{1}{4}$ cwt.

3. If there are 4 pulleys in the third system, and each weighs 2 lbs., what weight can be raised by a power equal to the weight of 20 lbs.?

4. A man weighing 15 stone, holding the bar to which the strings are attached in the third system of pulleys, can just raise a child weighing 1 stone, holding on to the last string. How many pulleys are there?

5. If there be three moveable pulleys, each of mass 1 lb., in a system in which all the strings are parallel and are attached to the weight, and the force required to support a certain weight is half that which would be required if the pulleys were weightless, find that weight.

6. Is it more advantageous in raising a weight by (i.) the first system, (ii.) the third system, of pulleys, to have the pulleys heavy or light?

7. If in a system of pulleys in which each hangs by a separate string there be three pulleys, each of mass 1 lb., and the power required to support a certain weight is twice that which would be required if these pulleys were weightless, find that weight.

8. Being given four weightless pulleys, find the greatest weights that can be supported by a force of 1 lb. when the pulleys are arranged as in the first, second, and third systems, respectively, in such a way that the effort is applied *downwards*.

9. If in Ex. 8 the effort of 1 lb. had to be applied *upwards*, what would be the weights lifted in the corresponding arrangements?

10. In the first system of pulleys, find from first principles the power necessary to support a weight of 4000 lbs. when there are four moveable pulleys. Find also the pull of each rope on the beam; and, if the sum is not equal to the weight, explain the difference.

11. If a man whose weight is 10 stone supports 2 cwt. by a block-and-tackle, there being 3 pulleys in the block, what is the pressure on the floor on which the man stands, and what the pressure on the beam to which the upper pulley-block is attached?

12. In the system of pulleys in which each pulley hangs by a separate string, how would you find experimentally the relation between the tension of any string and that of the string next above it?

13. Two bodies connected by means of a string passing over a smooth pulley touch each other at one point; show that the stress between them cannot be horizontal unless their weights are equal.

14. A man weighing 10 stone supports a weight of 91 lbs. by means of 3 moveable pulleys arranged in the first system and weighing, respectively, 2 lbs., 4 lbs., 5 lbs. What is the thrust of the man on the ground?

15. Apply the Principle of Energy to prove that in a system consisting of three very light pulleys, of which the first is fixed and each of the others is supported by the string of the preceding one, the strings being all parallel and all attached to the weight W, the downward acceleration of this weight, when the power P is not sufficient to balance it, is

$$\frac{W-7P}{W+49P}g.$$

CHAPTER IX.

COUPLES.—MACHINES: THE SCREW.

117. **A couple** consists of two forces of equal magnitude acting in opposite directions along two parallel straight lines (§ 76). *A couple cannot keep a body in equilibrium*, for it tends to rotate the body: the points of application of the two forces of the couple tending to move in opposite directions (§ 51). At the same time, the proof that two parallel forces have a single resultant fails for the case of a couple (§ 75).

Examples of couples.—In winding a clock we apply a *couple* to the key, for we do not try to make it move to one side or the other, but simply turn it round. To spin a small top between the finger and thumb, we apply a *couple* to it by moving the finger and thumb sharply in opposite directions. To open a door we apply a *couple* to the handle.

118. **Resultant of two parallel forces which nearly form a couple.**—Consider the resultant of two unlike parallel forces which are very nearly, but not quite, equal. Let P, Q be two such forces acting at A, B, and let $P > Q$. If R be the resultant of P and Q,

$$R = P - Q;$$

therefore, if Q is nearly equal to P, R is very small.

Also, if C be the point where the resultant meets AB produced,

$$AC \times R = BA \times P.$$

$$\therefore \quad BA = AC \times \frac{R}{P}$$

$$= AC \times \frac{P-Q}{P}.$$

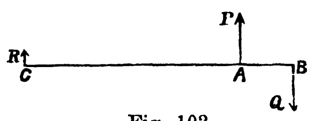


Fig. 102.

Since $P-Q$ is small, therefore BA is small in comparison with AC; hence AC is large in comparison with BA. Hence the resultant acts at a great distance from the line of action of either force. Writing the last equation,

$$AC = BA \times \frac{P}{P-Q},$$

we see that, if $Q = P$, the denominator becomes zero, and AC becomes infinitely great.

Hence *when two unlike parallel forces approach equality, and finally become equal, their resultant becomes infinitely small, and its line of action moves to an infinitely great distance from the components.*

For this reason it is convenient to consider the properties of couples apart from those of other systems of forces.

119. Definitions.—The **arm** of a couple is the perpendicular distance (AB. Fig. 103) between the lines of action of its two components (*i.e.*, the two forces forming the couple).

The **moment** of a couple is the algebraic sum of the moments of its two components about any point in their plane.

The following is the fundamental property of couples:—

120. The moment of a couple is the same about all points in its plane.

Let the couple consist of two equal and opposite forces P, $-P$ at B and A. Let O be any point in their plane. Draw OAB perpendicular to the forces.

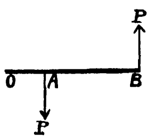


Fig. 103.

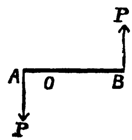


Fig. 104.

Then, if O does not lie between A and B, as in Fig. 104, we have

$$\text{algebraic sum of moments of forces} = P.OB - P\ OA$$
$$= P(OB - OA) = P.AB.$$

If O lies between A and B, as in Fig. 105, we have

$$\text{algebraic sum of moments} = P.OB + P.AO$$
$$= P(AO + OB) = P.AB.$$

Hence the moment of the couple about O is independent of the position of O and is equal to the product $P.AB$.

121. Cor. **Alternative expressions for the moment of a couple.**

The **moment** of a couple may therefore be defined as—

(i.) *The product of the measure of either force into the arm of the couple.*

(ii.) *The moment of either of the two forces about any point in the line of action of the other force.* For

moment $= P \times AB =$ moment about A of P acting at B.

122. **A couple cannot be replaced by a single finite resultant force.**

For the algebraic sum of the moments of two forces about any point on the line of action of their resultant is zero (§ 67). But the sum of the moments of two forces forming a couple about every point in their plane is a constant quantity, not zero.

Hence such forces cannot have a resultant. The same thing also follows from § 118.

*123. **A force, acting at any part of a body, is equivalent to an equal and parallel force acting at any other point together with a couple.**

Let P be a given force acting at any point O; to show that it is equivalent to an equal and parallel force P, acting at any other point A, together with a couple.

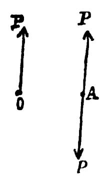


Fig. 105.

Introduce two equal and opposite forces P, $-P$, acting at A, numerically equal and parallel to the force P at O.

The effect of these forces will be to neutralize one another.

But the forces P at O and $-P$ at A form a couple whose moment (about A) is equal to the moment of the original force P about A.

Thus the original force P at O is equivalent to this couple and a parallel and equal force P at A.

124. **Two couples in the same plane whose moments are equal and opposite will balance one another.**

Let one of the couples consist of two forces P, $-P$ acting on the arm AB, and let the other couple consist of forces Q, $-Q$ acting on the arm CD.

Suppose, firstly, that P and Q are not parallel. Then the lines of action of the four forces must form a parallelogram $abcd$.

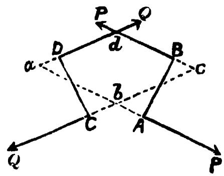


Fig. 106.

Since the moments of the couples are equal and opposite, and b lies on one of the forces of either couple, therefore the moments about b of P and Q acting along ad and cd are equal and opposite.

Therefore the resultant of P and Q at d passes through b, and therefore it acts along bd.

Similarly the resultant of $-P$, $-Q$ at b acts along db.

But the latter resultant is equal and opposite to the former, for the two components of the latter are respectively equal and opposite to those of the former.

Hence these two resultants balance each other, and therefore the four forces forming the two couples are in equilibrium.

To extend the proof to the case where the forces composing the two couples are parallel, it is only necessary to introduce two equal and opposite forces in a straight line, intersecting the forces P, $-P$ of one of the couples, and to compound one of these forces with P and the other with $-P$. We thus get a couple equivalent to the original one and of equal moment, but its component forces intersect those of the second couple; hence the above proof applies.

***125. A couple acting on a rigid body may be replaced by any other couple of equal moment acting on the body in the same plane without altering its effect.**

Let M be the moment of the given couple. Apply two couples of equal and opposite moments M, $-M$ to the body *anywhere* in the plane of the first couple. These balance each other, and do not therefore affect the body. Now combine the *first* and *third* couples. These also balance each other, since their moments (M, $-M$) are equal and opposite; therefore they may be removed. We are thus left with the *second* couple of moment M as the equivalent of the *first*.

Cor. From this result we see that a couple has no particular position of application, but that it may be shifted anywhere in its plane without altering its statical effect. The effect of the couple depends therefore only on its moment and the plane in which it acts.

Observation.—The above proof is identical in its reasoning with the proof of the Principle of the Transmission of Force. The present theorem may therefore be called the *Principle of the Transmission* (*or Transmissibility*) *of Couples*.

***126. To compound two coplanar couples into a single resultant couple.**—Let the couples consist of the forces (P, $-P$) and (Q, $-Q$) (Fig. 106), and let their lines of action be produced, if necessary, so as to form a parallelogram $abcd$.

Let R be the resultant of P and Q acting at d.

Then the resultant of $-P$, $-Q$ acting at b is evidently an equal and opposite force $-R$, parallel to R.

The forces R at d and $-R$ at b form a couple which is the required resultant of the two couples (P, $-P$) and (Q, $-Q$).

Since R is the resultant of P, Q acting at d,

$\therefore$ moment of R about b

= algebraic sum of moments of P, Q about b.

But b is a point on the line of action of the forces $-P$, $-Q$, $-R$.

Therefore the moments about b are the moments of the respective couples, and

$\therefore$ *moment of resultant couple*
= algebraic sum of moments of its component couples.

COR. *Any number of coplanar couples are equivalent to a single couple whose moment is the algebraic sum of their moments.* [For compound the couples together two at a time, &c.]

127. **Any number of coplanar forces are equivalent either to a single force or a couple, or are in equilibrium.**
Any two forces not forming a couple can be compounded together; and three or more forces cannot each form couples with all the others, for they cannot all be unlike. Therefore two of them must be capable of being replaced by a single force, and this process can be repeated until there are only two forces left. These either have a single resultant or form a couple or balance.

128. The screw.—Every one is familiar with a **screw.** It consists essentially of a cylindrical bolt OM, whose surface carries a *thread* or has a *groove* cut in it along a spiral curve. The form of this spiral can easily be constructed by taking a strip of paper with a straight edge, and wrapping it round a pencil in a slanting direction; the edge forms the curve like that along which the thread or groove runs.

The screw works in a *collar* or *nut* C, through which a hole is bored, having a groove to fit the thread or a thread to fit the groove of the screw.

When the screw turns in a fixed collar, it moves forward in the direction of its length. In each turn of the screw, the distance moved forward is equal to the distance between consecutive threads: *i.e.*, the distance DE between two consecutive turns of the thread, measured along the length of the bolt. This distance is called the **step.** Hence, by turning the screw round, it may be used to raise weights or overcome resistances applied to its end.

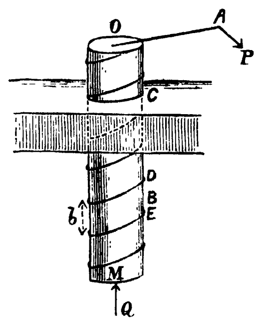


Fig. 107

The *effort* must tend to turn the screw, and must therefore have a *moment* about OM in a plane perpendicular to OM. Hence the effort may be a single force P applied at the end of a long arm OA, projecting at right angles to OM. More often the arm projects in both directions, as in the common screw press of Fig. 108, and two equal and opposite forces constituting a *couple* are then applied perpendicularly to its two extremities A, B.

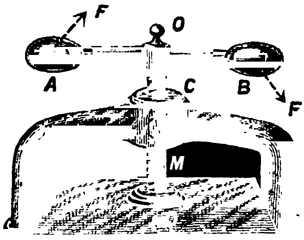


Fig. 108.

129. To find the mechanical advantage of a screw working without friction.

Let the effort P be applied perpendicularly at the end of an arm OA of length a, and let it overcome a resistance Q acting along the axis OM. Let b be the "step" or distance between two consecutive threads. Then, if the screw makes one complete turn, A the point of application of P will describe a circle of radius a about O, and will therefore move through a distance $2\pi a$ in the direction of P.* Also the screw will move through a distance b against the resistance Q.

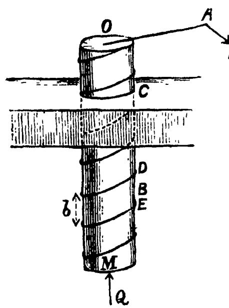


Fig. 109.

* As the screw moves forward along its axis OM, A really describes a spiral and not a circle; but, since P acts perpendicular to OM, the forward motion contributes nothing to the work done by P, which is therefore $P \times 2\pi a$.

Therefore, since

$$\text{work done by } P = \text{work done against } Q,$$

we have
$$P \times 2\pi a = Q \times b.$$

$$\therefore \textbf{ mechanical advantage } \frac{Q}{P} = \frac{2\pi a}{b} \quad \ldots\ldots\ldots\ldots\ldots \quad (1)$$

$$= \frac{\textbf{circumference of circle described by the arm}}{\textbf{step of screw}}.$$

COR. 1. Since the moment of P is $P \times a$, we have

$$\text{moment of } P = Q \times \frac{b}{2\pi}.$$

If the quantity $b/2\pi$ be called the **pitch** of the screw*
then **moment of effort = (resistance) × (pitch)** ... (**2**).

COR. 2. By the Principle of Moments the effort P may be replaced by any other force or couple whose moment tending to turn the screw is the same. If two forces F act in opposite directions at the ends of an arm AB of length $2a$ fixed at its middle point to the screw, the moment of the couple thus formed $= 2Fa$, and therefore

$$2Fa = Q \times b/2\pi,$$

a relation easily verified by the Principle of Work. This shows that the effect is the same as if the two forces F were both applied at one end of the arm, forming a single force $2F$.

Example.—A screw press is turned by applying forces of 21 lbs. in opposite directions to the ends of an arm 2 ft. long. If the step is $\frac{1}{8}$ in., to find in lbs. per sq. in. the pressure produced over the area of a circular piston 1 ft. in diameter.

Let Q be the total resistance. Then in one revolution of the screw the works done by the efforts and against Q are

$$2 \times 21 \times 2\pi \,.\, 1 \text{ ft.-lbs.} \quad \text{and} \quad Q \times \tfrac{1}{8} \times \tfrac{1}{12} \text{ ft-lbs.}$$

$$\therefore \quad \tfrac{1}{96} Q = 84\pi, \quad \text{or} \quad Q = 96 \times 84\pi.$$

Also $\quad$ area of piston $= \pi(\text{radius})^2 = \pi \,.\, 6^2 = 36\pi$ sq. ins.

$$\therefore \quad \text{required pressure} = \frac{Q}{36\pi} = \frac{96 \times 84}{36} = 224 \text{ lbs. per sq. in.}$$

$$= 2 \text{ cwt. per sq. in.}$$

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* This is the definition of "pitch" used by Sir Robert Ball in his "Theory of Screws," and now generally adopted, although in a number of books before us we find no less than *three* different definitions of *pitch*.

Summary of Results.

Couple = two equal unlike parallel forces. (§ 117.)

Moment of couple
= algebraic sum of moments of its forces about any point in its plane (§ 119.)
= moment of one force about point on the other (§ 121.)
= $P \cdot AB$ (where P = either force, AB = arm). (§ 120.)

Principle of transmission of couples.—A couple may be replaced by any other couple of equal moment. (§ 125.)

Resultant of two or more couples is couple whose moment = sum of moments of components. (§ 126.)

For a *smooth screw*, mechanical advantage

$$M \text{ or } \frac{Q}{P} = \frac{2\pi a}{b} = \frac{\text{circumference of circle described by } P}{\text{distance between two threads}} \quad \ldots (1). \quad (\S\ 129.)$$

$$\frac{\text{moment of } P}{Q} = \frac{b}{2\pi} = \frac{\text{step}}{2\pi} = \text{pitch of screw} \ \ldots\ldots (2).$$

EXAMPLES IX.

1. When a force and a couple act in the same plane on a rigid body, find their resultant.

2. Draw a square $ABCD$ and its diagonal AC. Two forces of 10 units act from A to B and from C to D respectively, forming a couple; a third force of 15 units acts from C to A. Find their resultant, and show in a diagram exactly how it acts.

3. Prove that a couple can be moved parallel to itself without altering its effect.

4. State (without proof) the conditions that must be satisfied in order that two couples may balance. Give a practical illustration of two balancing couples in different planes.

5. Forces P, Q, R act along the sides of a triangle ABC from B to C, C to A, and A to B, and are proportional to the lengths of the sides along which they act. (*a*) Show that they form a couple, and find its moment; (*b*) find their resultant when the direction of one of them (P) is reversed.

6. Forces P and Q act at A, and are completely represented by AB and AC, sides of a triangle ABC. Find a third force R, such that the three forces together may be equivalent to a couple whose moment is represented by half the area of the triangle.

7. The distance between two consecutive threads of a screw is $\frac{1}{4}$ in., and the length of the power arm is 5 ft. What weight will be sustained by a power of 1 lb.?

8. If a power of 10 lbs. acting on an arm 2 ft. long produces in the screw press a thrust of one ton weight, what is the "step" of the screw?

9. The arm of a screw-jack is 2 ft. long, and the screw rises 2 ins. when it is turned round nine times. What force must be applied to produce a thrust of half a ton weight?

10. Express the work done when a moment M has rotated n times. If a force equal to the weight of 10 lbs. revolve three times tangentially round a circle of 5 ft. radius, find the work it would do.

11. Show that any number of couples applied to a rigid body in one plane are equivalent to a single couple whose moment is the algebraic sum of the moments of the individual couples.

12. Forces act along the sides of a polygon, and are represented completely by those sides taken in order. Show that they are equivalent to a couple whose moment is measured by twice the area of the polygon.

13. In a weightless straight lever of the first order, show that in the case of equilibrium the power, the weight, and the reaction of the fulcrum form two unlike couples of equal moments.

14. A screw is formed upon a cylinder whose length is 1 ft. and circumference 3 ins. How many turns must be given to the thread in order that a power of 4 lbs. weight, applied tangentially at the edge of the cylinder, may support a weight of 2 cwt.?

EXAMINATION PAPER IV.

1. State the principle of Work, and apply it to find the relation between the effort and weight in the single-string system of pulleys, the strings being parallel and the number of pulleys n.

2. What is meant by a *couple* in Mechanics? Find the condition that two couples which act on a rigid body should equilibriate each other.

3. Prove that the mechanical advantage of n pulleys arranged in the separate-string system is 2^n, and arranged in the "third" system is $2^n - 1$; and draw the figure with five pulleys.

4. In the first system of pulleys, what weight will a power of 80 lbs. support if there are three moveable pulleys whose weights are 2, 3, and 4 lbs., respectively, the lightest being highest?

5. Find the relation of the power to the weight in the screw, neglecting friction.

6. In a screw-press a power equal to 16 lbs. weight, acting on an arm 3 ft. long, produces a pressure of half-a-ton. What is (i.) the "step," (ii.) the pitch, of the screw?

7. Explain why a rod which will support a considerable tension can be broken comparatively easily by being bent.

8. What is the resultant of a force of 4 lbs. weight and a couple with an arm 2 ft. long and forces 3 lbs. weight? State its position clearly with reference to the 4 lbs. force.

9. In the second system of pulleys, if a weight of 3 lbs. supports a weight of 15 lbs., and a weight of 5 lbs. supports a weight of 27 lbs., what is the weight of the lower block, and what would the mechanical advantage be if the lower block was weightless?

10. In the second system of pulleys, what must be the relation between the radii of the pulleys at the lower block in order that they may all be grooved in the same piece?

PART III.

CENTRES OF GRAVITY.

CHAPTER X.

CENTRES OF PARALLEL FORCES.

130. Method of finding the resultant of coplanar parallel forces.

When a number of parallel forces act on a rigid body, it would of course be possible to find their resultant by compounding two of them into a single resultant, then compounding this resultant with a third, and so on. But if the forces all act *in the same plane*, the same thing can be done more easily by writing down the equations which express the facts that—

(i.) *The* **magnitude** *of the resultant equals the algebraic sum of its components* (see § 131, below);

(ii.) *The* **moment** *of the resultant about any point equals the algebraic sum of the moments of its components.*

The point about which moments are taken may be chosen anywhere in the plane of the forces, but some points (very often *one* point) are generally more convenient than others. But it is important to notice that the final result is the same *whatever* point is chosen.

As it is advisable to assume *principles* rather than *formulæ* in all calculations, we subjoin the following example before giving a general investigation :—

Example.—Weights of 4 lbs. and 12 lbs. are attached to the ends of a uniform rod, 14 ft. long, weighing 12 lbs. To find the point at which it must be supported in order to balance.

The three forces on the rod are 4 lbs. and 12 lbs. acting at its ends, and its weight 12 lbs. acting at its middle point, 7 ft. from either end.

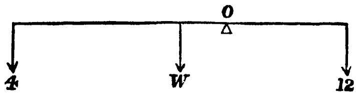

Fig. 110.

The *magnitude* of their resultant $= 12 + 12 + 4$ lbs. $= 28$ lbs.

The required point O is the point at which this resultant cuts the rod, and may be found in either of the following ways :—

(a) By taking moments about the end at which the 4 lbs. acts. Let the distance of O from this end be x ft. Then the equation of moments gives $28 \,.\, x = 4 \,.\, 0 + 12 \,.\, 7 + 12 \,.\, 14 = 252.$

$$\therefore \quad x = 9 \text{ ft.},$$

or the support is 9 ft. from the weight of 4 lbs., and 5 ft. from the other end.

(b) By taking moments about the other end. Let the distance of from that end be y. Then

$$28 \,.\, y = 12 \,.\, 0 + 12 \,.\, 7 + 4 \,.\, 14 = 140.$$

$$\therefore \quad y = 5 \text{ ft.},$$

or the support is 5 ft. from the 12 lbs. weight, agreeing with (a).

(c) By taking moments about the middle point. Let the distance of O from the middle point be z. Then

$$28 \,.\, z = 12 \,.\, 7 + 12 \,.\, 0 - 4 \,.\, 7 = 56.$$

$$\therefore \quad z = 2 \text{ ft.}$$

Hence the distances of O from the ends are $7 + 2$ ft. and $7 - 2$ ft.; *i.e.*, 9 and 5 ft., agreeing with (a) and (b).

(d) By taking moments *about O itself.* Let its distance from the middle point be z as before. Since the resultant passes through O, it has no moment about O. Therefore

$$\begin{aligned} 0 &= 12 \,.\, (7 - z) - 12 \,.\, z - 4 \,.\, (7 + z) \\ &= 84 - 12z - 12z - 28 - 4z = 56 - 28z; \end{aligned}$$

giving $$z = 2 \text{ ft.},$$

agreeing with (c). This last method does not require us to first find the magnitude of the resultant, but no *material* advantage is thereby gained.

131. To find the resultant of any number of given parallel forces P_1, P_2, P_3, ... acting in one plane.

Let P_1, P_2, P_3, ... denote the given forces both in magnitude and algebraic sign. Draw any line $OA_1A_2A_3 \ldots X$ at right angles to the forces. On it take any point O, the distances OA_1, OA_2 being known.

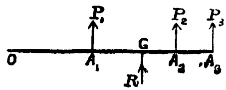

Fig. 111.

(i.) To find the magnitude of the resultant.

The resultant of P_1, P_2 is a parallel force of magnitude P_1+P_2; the resultant of this force and P_3 is therefore $P_1+P_2+P_3$; and so on. Hence the final resultant is a parallel force R, such that

$$R = P_1+P_2+P_3+\ldots \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$
$$= \text{algebraic sum of the forces.}$$

(ii.) To find the position of the resultant R.

Let it cut OX in a point G, the position of which is unknown. Since the moment of the resultant is equal to the algebraic sum of the moments of the components,

$$\therefore \quad R \cdot OG = P_1 \cdot OA_1 + P_2 \cdot OA_2 + P_3 \cdot OA_3 + \ldots .$$

Hence $$OG = \frac{P_1 \cdot OA_1 + P_2 \cdot OA_2 + P_3 \cdot OA_3 + \ldots}{R},$$

or $$OG = \frac{P_1 \cdot OA_1 + P_2 \cdot OA_2 + P_3 \cdot OA_3 + \ldots}{P_1 + P_2 + P_3 + \ldots} \ldots (2).$$

Equations (1), (2) determine, respectively, the magnitude and position of the resultant.

If x_1, x_2, x_3, ... denote the known distances of the component forces from O, and x the required distance of their resultant; equation (2) becomes $$x = \frac{P_1x_1 + P_2x_2 + P_3x_3 + \ldots}{P_1 + P_2 + P_3 + \ldots} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2a).$$

132. Equilibrium of a loaded beam resting on two supports.—When a beam is loaded with given weights placed at given points, and rests in a horizontal position on two props, it is often necessary to determine the thrusts supported by the props, or, what amounts to the same thing, the reactions of the props on the rod, which are equal and opposite to these thrusts.

These reactions, together with the weights on the beam, form a system in equilibrium, and therefore the sum of the moments of this system about any point is zero.

But we want to find the reactions one at a time, and therefore proceed as follows :—

(i.) *Take moments about one of the props. The reaction of that prop has no moment, and therefore the equation of moments at once gives the reaction of the other prop.*

We *might* now find the reaction of the first prop by taking moments above the second, but an easier plan is to

(ii.) *Equate to zero the algebraic sum of the forces (including the two reactions). The equation gives the sum of the reactions, and hence the other required reaction.*

Example.—A uniform rod, 12 ft. long, weighing 20 lbs., has weights of 12 lbs. and 4 lbs. attached to its ends, and 8 lbs. attached at a distance of 4 ft. from the 4-lb. weight. It is placed on two props, 8 ft. apart, so that the end with the 4-lb. weight projects 1 ft. To find the reactions of the props.

In Fig. 112, let M, N be the props, and let their reactions be R lbs. and S lbs., respectively.

Let G be the middle point of the rod at which its weight (20 lbs.) acts.

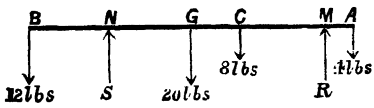

Fig. 112.

(i.) Taking moments about M, we have

$$S \cdot MN - 12 \cdot MB - 20 \cdot MG - 8 \cdot MC + 4 \cdot AM = 0,$$

or $$8S = 12 \cdot 11 + 20 \cdot 5 + 8 \cdot 3 - 4 \cdot 1$$

$$= 132 + 100 + 24 - 4 = 252\,;$$

whence $$S = 31\tfrac{1}{2} \text{ lbs.}$$

(ii.) Again, since the sum of the upward forces is equal and opposite to the sum of the downward ones (or the algebraic sum of the forces is zero),

$$\therefore \quad R+S-12-20-8-4 = 0,$$

or $$R+S = 12+20+8+4 = 44\,;$$

whence $$R = 44-31\tfrac{1}{2}\text{ lbs.} = 12\tfrac{1}{2}\text{ lbs.}$$

Hence the thrusts on the props are $12\frac{1}{2}$ lbs. and $31\frac{1}{2}$ lbs.

(iii.) If we had taken moments about N, we should have had

$$R\,.\,NM-4\,.\,NA-8\,.\,NC-20\,.\,NG+12\,.\,BN = 0,$$

$$8R = 4\,.\,9+8\,.\,5+3\,.\,20-12\,.\,3$$

$$= 36+40+60-36 = 100\,;$$

whence $$R = 12\tfrac{1}{2}\text{ lbs.},$$

agreeing with the value just found and affording a test of the accuracy of the calculation.

Observations.—The student will find it advisable, at first at any rate, to go through *all* the three processes [(i.), (ii.), (iii.)] illustrated above; in this way there will be a check on the accuracy of the work.

If, in any example, the thrust on one of the props should come out *negative*, it is to be inferred that the rod presses *upwards* on its support, and that the latter has to hold it down.

[The general formula is obtained as follows :—

Let a beam, whose weight is W, acting at its centre of gravity G, be supported at M, N, and let any weights w_1, w_2, w_3 be attached at points A_1, A_2, A_3. Then, if R, S denote the unknown reactions at M, N, the equation of moments about N gives

$$R\,.\,NM = w_1\,.\,NA_1+w_2\,.\,NA_2+w_3\,.\,NA_3+W.\,NG\,;$$

and the equation of moments about M gives

$$S\,.\,MN = w_1\,.\,MA_1+w_2\,.\,MA_2+w_3\,.\,MA_3+W.\,MG.$$

By addition, $(R+S)\,MN = w_1\,.\,MN+w_2\,.\,MN+w_3\,.\,MN+W.\,MN,$

or $$R+S = w_1+w_2+w_3+W.$$

This is the equation which we should obtain by resolving perpendicular to the rod, or equating the algebraic sum of the forces to zero, showing that the values of R, S found by resolving and taking moments about the supports are consistent.]

133. **Conditions of equilibrium of a heavy lever.**— When the weight of a lever itself has to be taken into account, the condition of equilibrium may be found as in other cases by taking moments about the fulcrum.

Let a lever, whose weight w acts at G and whose fulcrum is C, be used to overcome a resistance or raise a weight W applied at B by means of an effort applied

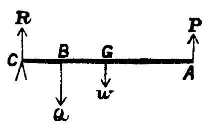

Fig. 113.

at A. Then, if P denote this effort (considered positive when acting upwards), the equation of moments about C gives $$P \cdot CA = W \cdot CB + w \cdot CG \quad\text{..............}(3);$$ and, if the lengths CA, CB, CG are considered positive or negative according to the direction in which they are drawn from C, this formula will be applicable to levers of any class.

The upward reaction (R) at the fulcrum will be given algebraically by $$R + P = W + w.$$

Observation.—These formulæ hold good *whether the lever be horizontal or inclined to the horizon*, provided that the forces on the lever are parallel, and that their points of application lie in a straight line through the fulcrum. This will be proved in § 138.

Example.—A beam, whose weight is W, acting at its middle point, can be tilted up with one end remaining on the ground by a force $\frac{1}{2}W$ applied at the other end. If a prop be placed under the end that has been raised, the end on the ground can now be raised by a force $\frac{1}{2}W$; thus the whole beam can be raised with the same mechanical advantage that is obtainable with a single moveable pulley.

The work done is (as it should be) the same as if the beam were lifted bodily up; for each end in turn is raised through the whole height risen, so that the distance moved by the effort is *twice* this height.

134. To find the resultant of any number of parallel forces of given magnitudes applied at given points of a rigid body, not necessarily in the same plane.

Let the given parallel forces be P acting at A, Q at B, R at C, S at D, and so on.

Join AB, and in AB take a point E such that
$$P \times AE = Q \times EB.$$

Then, by § 77, the forces P and Q are equivalent to a single resultant force $P + Q$ acting at E, parallel to both of them.

Now compound this resultant with the force R at C. Join EC, and on it take a point F, such that

$$(P+Q)\times EF = R\times FC\ ;$$

then the forces $P+Q$ at G and R at C are equivalent to a single resultant force $P+Q+R$ acting at F parallel to them.

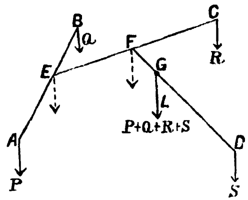

Fig. 114.

This resultant is therefore the resultant of the *three* forces P, Q, R at A, B, C, respectively.

Now join FD, and in it take a point G, such that

then the forces $P+Q+R$ at F and S at D are equivalent to a single resultant $P+Q+R+S$ acting at G parallel to them.

This resultant is therefore the resultant of the *four* forces P, Q, R, S, at A, B, C, D, respectively.

Proceeding in this way, we may find the resultant of any number of parallel forces acting at different points of a body.

We might have compounded the four forces P, Q, R, S, in any other order whatever. Thus, if we had first found the resultant of R and S, then compounded this with Q, and compounded the resultant so obtained with P, we should have a different construction for the resultant of the four forces.

135. The Centre of Parallel Forces.—DEFINITION.—The above constructions show that—

The resultant of any number of parallel forces passes through a certain point whose position depends only on the magnitudes and the points of application of the forces.

This point is called the **centre** *of the parallel forces.*

Thus, in § 134, *E* is the centre of *P* and *Q*, *F* is the centre of *P, Q, R*, and *G* is the centre of *P, Q, R, S*.

If the forces *P, Q, R, S*, instead of acting as in Fig. 114, are applied at the same points *A, B, C, D*, but in a

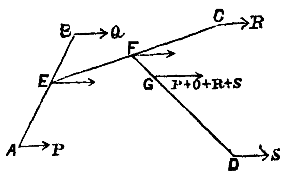

Fig. 115.

different direction, as in Fig. 115, their resultant will still pass through the same point *G* as before.

OBSERVATION.—The student should carefully notice the difference between the *resultant* and the *centre* of a system of parallel forces. If the forces *P, Q, R, S* act at *A, B, C, D in a given direction*, their resultant acts in a straight line *GL*, *parallel to this direction and passing through the centre G*. The resultant *may be supposed to act at G*, but it *need not necessarily be applied* **at** *G*; for, by the Principle of Transmission of Force, it may be applied at any point (say *L*) in its line of action (Fig. 114). But let *P, Q, R, S*, still acting at *A, B, C, D*, be turned round into a different direction. Their resultant will still pass through *G*, the centre of the forces, but it will no longer pass through *L* (Fig. 115).

Hence the resultant may always be supposed to act at *G*, no matter what the direction of the forces may be; but it can only be supposed to act at any other point *L* if its line of action passes through *L*, and this requires the forces to be parallel to *GL*.

136. **A system of parallel forces cannot have more than one centre.**—This may be shown by *reductio ad absurdum* thus:—Suppose the forces P, Q, R, S at A, B, C, D to have two centres G, H. Take the forces in any direction not parallel to GH. Then a system of such forces which may have G for their centre will be balanced by a system of equal and opposite forces which may have H for their centre; hence the two resultants acting at G, H, and forming a couple (for they do not act in the line GH) will keep the body in equilibrium, which is impossible

137. **Centre of parallel forces acting at points in a straight line.**—If a number of parallel forces P_1, P_2, P_3 are applied at points A_1, A_2, A_3 lying along a straight rod or beam, the construction for the centre of parallel forces (§ 134) shows that their centre also lies in the straight line through A_1, A_2, A_3. If x_1, x_2, x_3 are the distances of A_1, A_2, A_3 from a fixed point O on the rod, the distance of the centre from O is therefore given by the formula of § 131,

$$x = \frac{P_1x_1 + P_2x_2 + P_3x_3 + \dots}{P_1 + P_2 + P_3 + \dots}.$$

This follows at once, by taking moments, if we suppose the forces applied perpendicular to the rod. If they act in any other direction the products P_1x_1, P_2x_2, P_3x_3, ... will not be the moments of the forces about O; but, since the centre of the parallel forces is unaltered in position, the formula shows that it may be found *by treating the products P_1x_1, P_2x_2, P_3x_3, ... as if they were moments.* In other words, *we may replace the arms on which the parallel forces act about O by their distances from O measured along any straight line cutting the forces.*

138. **Theorem.**—*If any number of weights balance when attached to a straight lever placed horizontally, they will continue to balance when the lever is turned round about its fulcrum into any direction whatever.*

For the weights are parallel forces acting at points in a straight line through the fulcrum, and their resultant is balanced by the reaction of the fulcrum. Hence the centre of the parallel forces is at the fulcrum. If the forces on the lever were altered in direction (still remaining parallel), their resultant would, therefore, still be balanced by the reaction of the fulcrum, and the effect is evidently the same if the lever itself be turned round instead and the weights continue to act vertically.

Summary of Results.

If a number of parallel forces P_1, P_2, P_3, &c., cut a straight line at distances x_1, x_2, x_3, &c., from any point O, their resultant is given by the equation

$$R = P_1 + P_2 + P_3 + \dots \qquad \dots\dots\dots\dots\dots (1),$$

and its distance x from O by

$$x = \frac{P_1 x_1 + P_2 x_2 + P_3 x_3 + \dots}{P_1 + P_2 + P_3 + \dots} \qquad \dots\dots\dots\dots (2),$$

provided the proper sign is given to each force and distance. (§ 131.)

The centre of parallel forces P, Q, R, S at A, B, C, D, &c., is found by dividing

AB at E, where $P \times AE = Q \times EB$;

EC at F, where $(P+Q)\,EF = R \times FC$;

FD at G, where $(P+Q+R)\,FG = S \times GD$;

and so on. (§§ 134, 135.)

EXAMPLES X.

1. Two men carry a load of $\frac{3}{4}$ cwt. suspended from a horizontal pole 14 ft. long, whose weight is 10 lbs., and whose ends rest on their shoulders. Find the point at which the load must be suspended in order that one of the men may bear 71 lbs. of the whole weight.

2. Three like parallel forces of 5 lbs., 7 lbs., and 9 lbs. act in lines whose distances apart are 3 ft. and 4 ft. Find the resultant.

3. A uniform beam, 14 ft. long and weighing 120 lbs., is attached to two props, one of which is 3 ft. and the other 5 ft. from its centre. Calculate the forces on the props when a weight of 100 lbs. is placed first at one end and then at the other end of the beam.

4. A uniform bar, 3 ft. long and weighing 5 lbs., rests on a horizontal table with one end projecting 4 ins. over the edge. Find the greatest weight that can be hung on the end without making the bar topple over.

5. If a beam 6 ft. long, and weighing 15 lbs., is acted upon by a downward force of 3 lbs. at one end and an upward force of 7 lbs. at the other end, what is required to keep it in equilibrium in a horizontal position?

6. A bar of uniform thickness and density, 12 ft. long and 1 cwt., is supported at its extremities in a horizontal position. If a body of 2 cwt. be suspended from a point 2 ft. distant from one end, and a body of 4 cwt. at 4 ft. from the other end, required the pressures on the points of support.

7. A uniform beam, 12 ft. long and weighing 72 lbs., is supported on two props, 1 ft. and ·2 ft., respectively, from the ends. Where must a weight of 24 lbs. be hung to make the pressure on each prop equal to 48 lbs.?

8. A rod without weight rests horizontally on two points, A and B, 10 ft. apart. Between A and B take points C, O, D, such that $AC = 2$ ft., $AO = 4$ ft., $AD = 7$ ft. A weight of 100 lbs. is hung at C, and one of 90 lbs. at D. Find the algebraical sum of the moments with respect to O of the forces on one side of O.

9. A uniform beam, 12 ft. long and weighing 56 lbs., rests on, and is fastened to, two props 5 ft. apart, one of which is 3 ft. from one end of the beam. A load of 35 lbs. is placed (a) on the middle of the beam, (b) at the end nearest a prop, (c) at the end furthest from a prop. Calculate the weight each prop has to bear in each case.

10. Equal parallel forces act at the angular points of an equilateral triangle. Find their centre (i.) when the forces are like, (ii.) when one of them is unlike the other two.

11. P, Q, R are parallel forces acting in the same direction at the angular points of an equilateral triangle ABC. If $P = 2Q = 3R$, find the position of their centre.

12. In the preceding question, find the position of the centre of the forces if the direction of Q is reversed.

13. A rod is supported horizontally on two points A and B, 12 ft. apart. Between A and B points C and D are taken, such that $AC = BD = 3$ ft. A weight of 120 lbs. is hung at C, and a weight of 240 lbs. at D; the weight of the rod is neglected. Take a point O

midway between A and B, and find with respect to O the algebraical sum of the moments of the forces acting on the rod on one side of O.

14. Parallel forces of 3, 5, 7, and 11 lbs. act at points in a straight line at distances of 4 ins. from each other. What is the distance of their centre from the 5 lb. force?

15. A uniform straight lever of weight 10 lbs. and length 6 ft. balances about a certain point when weights of 3 lbs. and 5 lbs. are attached to its extremities. How far must the point of support be moved so that the lever may balance when each of the weights is increased by 1 lb.?

16. Four equal forces act at the corners of a square in the same direction along parallel lines. What change will be produced in the position of their centre if the direction of one of them is reversed?

17. If a series of parallel forces can be reduced to a couple, what is the position of their centre?

18. $ABCD$ is a square, and AC a diagonal. Forces P, Q, R act along parallel lines at B, C, D, respectively; Q acts in the direction A to C, P and R act in the opposite direction. Find, and show in a diagram, the position of the centre when $Q = 5P$ and $R = 7P$.

19. $ABCD$ is a square table whose side is 3 ft. long. The sides AB, BC, &c., are bisected at E, F, G, H, respectively, and O is the centre of the table. At A, B, C, D, O act upward parallel forces of 1 lb., 2 lbs., 3 lbs., 4 lbs., and 5 lbs., respectively, and at E, F, G, H act downward parallel forces of 2 lbs., 1 lb., 2 lbs., 1 lb., respectively. What is the position of the centre of the nine forces?

20. In the third system of pulleys, a weight of 45 lbs. is supported by a force of 3 lbs. weight. The strings are attached to a 6-ft. bar from which the weight is suspended, and at distances of 1 ft. from each other. To what point of the bar should the weight be attached? The bar and pulleys may be considered weightless.

CHAPTER XI.

DEFINITIONS AND PROPERTIES OF THE CENTRE OF GRAVITY.

139. **Every body has a centre of gravity.**—We have frequently made use of the property that the weight of a heavy straight uniform rod or beam may be supposed to act at its middle point, and, generally, that the weight of a rigid body of any shape may be supposed to be concentrated at a single point, called the centre of gravity of the body. We shall now prove this property, and in Chap. XII. we shall show how to determine the position of this point for bodies of certain shapes.

Gravity.—Defining *gravity** as the attraction which the Earth exerts on all bodies, and the *weight* of a body as the force with which that body is attracted to the Earth, it is shown in *Dynamics*, Chap. VIII., that, at any given place the weights of different bodies are proportional to their masses (*i.e.*, the quantities of matter in them), and that the weight of a body in pounds' weight measures the mass of the body in pounds.

The weight of a body always tends to pull it towards the centre of the Earth. Hence, defining the *vertical* as the direction of gravity, the verticals at different places would, if produced, meet in the Earth's centre. But the Earth's radius is nearly 4000 miles; consequently the verticals at two places would have to be produced nearly

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* We are here dealing with *terrestrial gravity* only. Considerations respecting the *universal gravitation* which exists between all bodies are beyond the scope of this book. We omit the subject altogether in preference to furnishing a few fragmentary statements about it, and refer the advanced student to Barlow and Bryan's *Astronomy*, Chap. XIII., Sections II. and III.

4000 miles below the Earth's surface before they would meet. Hence, unless the places themselves are at a considerable distance apart, the verticals are very approximately parallel, and we shall consider them as parallel in treating of the centre of gravity.

Now suppose a body to be built up of a number of heavy particles rigidly connected together. Then, unless the body is very large (so large, indeed, as to be comparable in size with the whole Earth), the weights of the particles will form a system of parallel forces acting on them. By § 135, these forces will have a centre through which their resultant always passes. This centre is called the **centre of gravity** of the system of particles.

The same thing is true for any body, whatever be its nature or the distribution of its parts. For if we subdivide any body into a very large number of parts, we may make these parts so small that each may (for all practical purposes) be regarded as a single heavy particle. Hence the resultant weight due to the weights of the different portions of a body always passes through a certain point in the body, and this point is called the **centre of gravity** of the body.

The centre of a system of parallel forces was defined by the property that the resultant force continues to pass through this point, even when the direction of the forces is changed, provided that they remain parallel forces and have the same magnitudes as before. We cannot alter the direction of gravity, but it will amount to the same thing if, instead, we turn the body itself round, provided that in doing so we do not alter its size and shape, and that we suppose the centre of gravity to move as if rigidly connected with it.

The centre of gravity of a body is therefore *fixed relative to the body*, so that, when the body moves as a rigid body, the centre of gravity moves with it.

We may, if we please, suppose the body to have a small speck in it, marking the position of the centre of gravity. However the body is moved, its weight always acts in a straight line through this speck.

It is not necessary that the body itself should be rigid in order to have a centre of gravity, but the centre of

gravity remains fixed relative to the body only as long as its size and shape remain unaltered.

Thus a straight piece of wire has a centre of gravity at its middle point, and its weight will act at that point as long as it remains straight. If the wire be bent, it will no longer have the same centre of gravity. A bicycle has a centre of gravity through which its weight will always act, so long as its different parts (the steering-wheel and handle, and the frame, &c.) preserve the same relative position.

140. We may therefore give the following :—

DEFINITION.—The **centre of gravity** of a body is that point, fixed relative to the body, through which the resultant force due to the Earth's attraction on it always passes, whatever be the position of the body, provided that its size and shape remain constant.

In short, *the centre of gravity is the point at which the whole* **weight** *of the body always acts.*

The abbreviation for centre of gravity is **C.G.**

141. **The construction for the C.G. of a system of particles** at given points is identical with that for the centre of parallel forces, the weights of the particles simply taking the place of the parallel forces. It is, therefore, given by § 134, only that, instead of replacing the *forces* two at a time by a single resultant *force*, we have to replace the *weights* two at a time by a single *weight.**

142. **A body cannot have more than one C.G.**—For if the body had two centres of gravity G, H, the line of action of the body's weight would always have to pass through both G and H, and would therefore have to be the line GH. But this is impossible, except when GH is vertical, for the weight of a body always acts vertically. Hence the body cannot have two C.G.'s.

* The student is recommended to write out § 134, substituting the words "weight" for "parallel force," and "centre of gravity" for "centre of parallel forces."

143. Centre of mass.—The foregoing considerations show that a body has a C.G. only when—

(i.) It is acted on by gravity, *i.e.*, it has *weight.*

(ii.) The intensity of gravity is constant, both in magnitude and direction, at all points of the body (as is the case for bodies of small size as compared with the Earth).

Now, if gravity were not to exist, a body would not have *weight*, but it would still have what is called *mass*, although its mass could not be measured in the ordinary way by *weighing*, and the analogy naturally suggests (as we shall now learn) that the body would still have a *centre of mass.*

Suppose a body to be at first acted on by gravity, and let its C.G. be found and indicated in position by marks made on the body. Suppose the body to be then removed from the influence of gravity by being carried right away from the Earth into space. The marks will still indicate the same point of the body, and the position of this point will have a definite relation to the distribution of matter in the body, although it can no longer be called its centre of *gravity*. This point will be called the body's *centre of mass*. It may be defined *statically* as follows:—

144. DEFINITION. — The **centre of mass,** or **mass-centre,** of a body is the centre of a system of parallel forces acting on all the particles into which the body may be supposed divided, and such that the force on each particle is proportional to its mass.

The abbreviation for centre of mass is **C.M.**

We observe that every body, or system of bodies, has always a C.M.* *If* it have a C.G., that point will *necessarily* coincide with its C.M., because the weights of the particles of the body, being proportional to their masses, form a system of forces having the C.M. for their centre.

It is, therefore, always correct to speak of the C.G. of a body as its centre of mass, but not *vice versâ.*

As we are only concerned with the effects of terrestrial

* For the definition makes no assumption as to the *cause* producing the forces on the particles of the body, and only assumes that these particles have *mass.*

gravity, we shall continue to use the term C.G. By finding the C.G., we really determine the C.M.; just as by *weighing* a body we really determine its *mass*.

[* The following *dynamical* definition may be substituted for the above equivalent statical definition:—

DEFINITION.—The **centre of mass** of a body is the point through which the resultant ("impressed") force on the body acts when all parts of the body move with the same acceleration.*

For the forces acting on different parts of the body are proportional to their respective masses, as shown in *Dynamics* (Chap. VI., § 83).]

[* 145. **Construction for the C.M. of a number of particles.**—If particles, whose *masses* are P, Q, R, S, ..., are situated at points A, B, C, D, the point G, obtained by the construction of § 134, will be the centre of parallel forces of magnitudes P, Q, R, S acting at A, B, C, D, respectively, and will therefore be the C.M. of the particles.]

146. DEFINITIONS. — A **lamina** is a sheet of material whose thickness is so small that it may be regarded as a distribution of matter over an area. A **uniform** lamina is one which is of the same thickness and formed of the same substance throughout.

The C.G. of an area or surface means the C.G. of a uniform lamina covering that area or surface.

Thus a sheet of paper or thin card, a thin sheet of metal such as that forming a tin canister, a plate of glass of small thickness, a slate, and in some cases even a wooden plank, may be regarded as a lamina. And the C.G. of a parallelogram will be the C.G. of a sheet of paper or any other uniform lamina forming that parallelogram.

A wire, stick, rod, or beam is said to be **uniform** when it is of the same cross section and of the same substance throughout its length.

The C.G. of a line (whether straight or curved) means the C.G. of a thin uniform wire placed along that line.

* This is the case when the motion of the body is one of "pure translation."

147. The C.G. of a straight line is at its middle point.

This is obvious, for there is no reason why it should be nearer one end than the other.

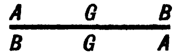

Fig. 116.

To prove it, let AB be a straight uniform wire, G its C.G. Then G evidently lies somewhere in the line AB. And if the wire be turned round so that the end B is placed at A and the end A is placed at B, the wire lies along the same line as before, and the position of G must therefore be unaltered. Hence $GA = GB$, or G is the middle point of AB.

148. The C.G. of

(*a*) a circular ring, and

(*b*) a circular area,

are at the centre of the circle itself.

For, if either a uniform circular wire or a uniform circular lamina is rotated about its centre, it will continue to occupy the same space as before; therefore its C.G. will be unaltered in position. But if the C.G. were not at the centre of the circle, its position would rotate with the wire or lamina, and would therefore change. Therefore the C.G. cannot be elsewhere than at the centre (as is almost obvious without proof).

149. **Other symmetrical figures.** — The method of the last three articles depends on the fact of the figure whose C.G. is required being "**symmetrical** about a point," in such a way that it is *capable of being turned about that point till it occupies exactly the same space as before, without its different parts all coming round to their original positions.* We find, in each case, that *one* point alone remains unchanged in position, and we conclude that this point must be the required C.G.

A body of *any* shape could, of course, be *rotated completely round* (*i.e.*, turned through four right angles) about any point and brought back into its original position; but this would *not* help us to find the C.G., because *every* point of the body would come back to its former

position, so that we should be no nearer knowing *which* was the C.G. than we were before.

From the above property we are able at once to write down the following additional results:—

The C.G. of the area of a **regular polygon** is the centre of the polygon, and the same is true for a uniform wire bent round to form the sides or "perimeter" of a regular polygon. For the C.G. must be unaltered in position when the polygon is turned about the centre, and each side is brought into the former place of the next side.

The C.G. of a solid **sphere,** or of the surface of a sphere (Fig. 117), is at O, the centre of the sphere, for its position

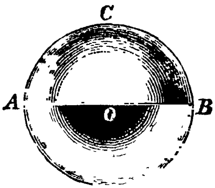

Fig. 117.

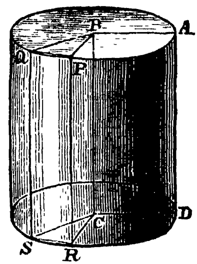

Fig. 118.

is evidently unaltered by turning the sphere round about its centre through any angle.

The C.G. of the volume or the surface of a **right cylinder** (Fig. 118) is at the middle point of its axis BC, for it is evidently unaltered in position when the cylinder is revolved about its axis, and also when turned, so that the positions of the two ends B, C are interchanged.

The C.G. of the volume or surface of a **cube** or any other **rectangular parallelepiped** is at a point midway between each pair of opposite faces, as may be seen by turning it round so as to interchange the positions of two opposite faces. It may be proved geometrically that the diagonals all intersect and bisect one another at the C.G.

150. Properties of the C.G. — Equilibrium of a heavy body about a fixed point.

From what we have learned in the present chapter, we know that, *if a heavy body* (*i.e.*, a body on which gravity can act) *be supported at any point, the* C.G. *and the point of support will be in the same vertical line*, for the only forces acting on the body are—

(i.) The resultant force due to gravity. This is the weight of the body, and it acts vertically through the C.G.

(ii.) The reaction at the point of support.

These forces are in equilibrium; therefore they must be equal and opposite and in the same straight line.

Therefore the point of suspension is in the vertical through the C.G.

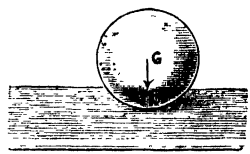

Fig. 119.

Theoretically, a body would balance when supported at a point either vertically *above* its C.G. (Fig. 120), or vertically *below* its C.G. (Fig. 119), or *at* its C.G. But it sometimes happens (as we shall see later on) that a body so supported is "top heavy," and it is *practically* impossible to keep it balanced any length of time.

151. If a heavy body is supported *at* its C.G., it will balance in every position.

Since the C.G. is at the point of support, the body's weight always acts through the point of support. This weight cannot move the point of support, because that point is fixed; and it cannot turn the body round, because it has no moment about that point. Hence the body must remain balanced in every position.

In consequence of this property, the C.G. *is very commonly spoken of as the* **balancing point** *of a body.*

152. **The plumb-line,*** used by builders and others for finding the direction of the vertical, is a flexible string from which a heavy weight of lead, or **plummet,** hangs. When at rest, the string is vertical throughout its length,

* From the Latin *plumbum* = "lead."

for the tension at any point has only to support the vertical forces due to the weight of the plummet and the part of the string below that point. Hence the tension is everywhere vertical. And the string, being flexible, places itself so that the tension is everywhere along the string; hence the plumb-line is itself vertical.

153. To find by experiment the C.G. of a lamina.

The C.G. of a sheet of metal or any other material, or a wooden plank, or a wire of any shape, may be found in the following manner:—

Suspend the body from any point A, and on it draw the vertical line AD through A. This line may be traced either by means of a plumb-line or by producing the direction of the string from which the body hangs. We know, by § 150, that the C.G. lies in AD.

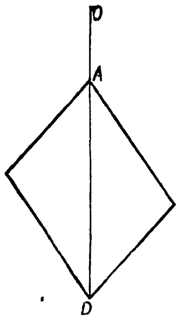

Fig. 120.

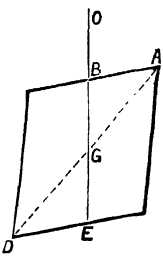

Fig. 121.

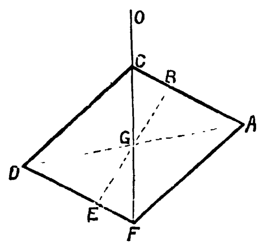

Fig. 122.

Now suspend the body from any other point B, and on it draw the vertical line BE through B. As before, the C.G. lies in BE. Therefore the required C.G. is at the intersection of AD, BE.

This method is *theoretically* applicable to any body whatever, but except in the case of a lamina, it would usually not be very easy to mark the two positions of the vertical, AD and BE.

154. As an instructive exercise, the student should perform a few simple experiments, such as the following:—

EXPERIMENT I. — Take a sheet of cardboard, wood, or metal, hang it in succession from two points A, B, and each time mark on it the vertical through the point of suspension. Let these lines be AD and BE, and let them intersect in G. Then, by placing a finger under the lamina at G, it will be found possible to balance it in a horizontal position, showing that G is its balancing point or C.G. Now hang the lamina from a third point C. It will be found that G is vertically below C, thus verifying that the C.G. and the point of suspension always lie in the same vertical line.

Now let a small hole be drilled through the lamina at G, and let it be supported in a vertical plane by a thin nail or pin passing through this hole. If the position of G has been found with accuracy, the body can be turned round about the pin and will remain balanced in every position.

If the lamina is in the shape of a rectangle or parallelogram, and is hung up from one of its angles, the diagonal through that angle will be found to be vertical, thus verifying that the C.G. lies in the diagonals, as in Figs. 120, 122.

EXPERIMENT II. — Take a straight thin rod (*e.g.*, a ruler, a straight walking-stick, or, better still, a billiard cue), and suspend it by means of two very fine strings from any point O. Also make a plumb-line by hanging a small weight W by a single thread from O. If the thread OW cuts the rod in G, the C.G. of the rod will be at G. Mark this point on the rod. Then, if the rod be laid on the finger or on any support touching it at G, it will balance.

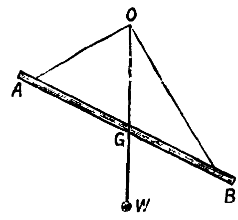

Fig. 123.

If the rod is of uniform thickness and material throughout, G will be found to be at its middle point. If one end of the rod is heavier than the other, G will be found to be nearer to the heavier end.

EXPERIMENT III.—Take a heavy piece of wire bent into a hoop or any other form, and suspend it in turn from different points A, B, C on it. By means of a plumb-line, mark the points D, E, F on the wire, which are vertically below A, B, C in the respective positions. Then, by placing the wire on a sheet of paper, the lines AD, BE, CF may be drawn, and it will thus be verified that these lines all pass through one point G. This point is the C.G. of the wire, *and is usually not on the wire itself.* In fact, if the wire forms a circular hoop, its C.G. will be at the centre of the circle, and therefore it will never be on the hoop.

155. DEFINITION.—When a body rests upon any hard flat surface, its **base** is defined to be the area enclosed by a fine string drawn tightly round it so as to enclose all points of the body in contact with the supporting surface.

When a glass tumbler rests on a table, the *base* of the tumbler is evidently the circular area enclosed by the parts of the tumbler touching the table. But in the case of a table resting on the floor on its four legs, the *base* is the quadrilateral formed by joining the feet, for this would be the figure assumed by a string pulled tightly round the points of contact. The definition ensures that when the body is overturned it must turn about some point or line bounding the base.

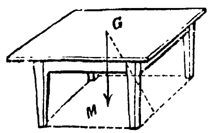

Fig. 124.

[The base may also be defined as "the smallest *non-re-entrant* figure which contains every point and surface of contact."]

156. Equilibrium of a heavy body resting on its base.

With the definition given above, we shall now show that

When a body is placed on a plane it will stand or fall according as the vertical line through its C.G. falls within or without the base on which it rests.

(i) First let the vertical through the C.G. cut the supporting plane at a point M *outside* the base AD; let A be the point of the base that is nearest to M. Then the

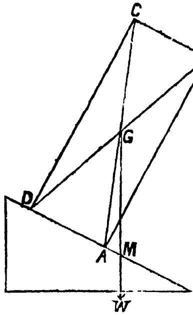

Fig. 125.

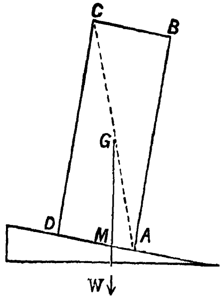

Fig. 126.

weight of the body is equivalent to a single resultant force acting at G, and the moment of this weight (Fig. 125) tends to turn the body about A in such a way as to lift up all the other points of the body touching the plane. Now the reaction of the plane always acts away from it, and never tends to prevent any part of the body from being lifted off. Hence there is nothing to counteract the tendency of the body to overturn about A. Therefore it will *fall.*

(ii.) Second, let the vertical through the C.G. cut the supporting plane at a point M inside the base AD. Then, if A be any point at the edge of the base, the moment of the weight about A tends to press down the other points of the body touching the plane (Fig. 126), as at D. Now the reaction of the plane prevents the body from penetrating it. Hence, in order to overturn the body about A, we should therefore have to turn it in the *opposite* direction to that in which its weight tends to turn it. Therefore the body will *stand.* And if it be slightly tilted up, its weight will bring it back to its original position as soon as it is let go.

As a particular case, when a body rests touching the

ground at a single point, the vertical through the centre of gravity must pass through that point (Fig. 119).

Observation.—The plane supporting the body may be either horizontal or inclined, provided that it is sufficiently rough to prevent the body from sliding down.

157. **Experimental verifications.** — The reader is strongly recommended to test the truth of the theorem by some simple experiments made with any common objects around him. The following are a few examples of its application:—

Experiment I.—A plate or saucer may be placed overlapping the edge of a table, provided that its centre (or C.G.) does not project beyond the edge (or *base*). If the centre overlaps, the plate will fall. By placing something heavy on the portion of the plate *on* the table, the position of the C.G. of the whole may be brought nearer the table, and the plate may then be made to rest overlapping beyond its centre, but not beyond the new C.G.

Similarly, a book can be placed on the top of another book resting on a table, provided that the middle (or C.G.) of the upper book does not project beyond the lower book (*i.e.*, beyond the *base*). Here, again, to make the upper book project beyond its middle, weights must be placed on the supported end to bring the C.G. nearer that end.

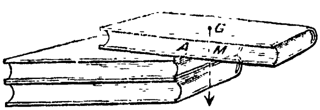

Fig. 127.

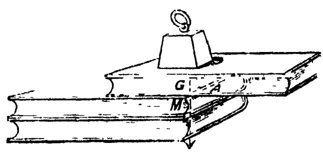

Fig. 128.

Experiment II.—If any body of rectangular section *ABCD* (say a brick) be stood upon a rough plank, and the plank be gradually tilted about the edge *A*, the body will remain standing as long as its C.G. does not overlap the base, but directly this happens it will fall over about its lowest edge (Figs. 125, 126).

Now the C.G. of a rectangular body lies in the diagonal

plane AC. Hence the body will turn over just after the diagonal AC has passed through the vertical position.

158. **Other illustrations.**—(1) A cart or tricycle will overturn if the vertical through its C.G. falls outside the wheel base (Fig. 129).

Fig. 129.

(2) A porter carrying a heavy trunk in one hand often extends the opposite arm at full length in order to more readily bring his C.G. over a point between his two feet. A man carrying a heavy weight in front of him leans back in order to bring his C.G. over his base.

(3) If we lean too far back in a chair tilted up on its hind legs, we shall fall over backwards when the vertical through the C.G. falls behind the line joining the two back feet.

(4) The leaning tower of Pisa overlaps it base by 13½ feet at its highest point, but it stands because the vertical through its C G. is well within the base. Treating the tower as a uniform cylinder (Fig. 126), its C.G. is at G midway between two diagonally opposite points A, C at the top and bottom. Hence the tower would fall if it were to project beyond A, that is, if the top were to overlap by more than the diameter of its base.

159.* **Stable, unstable, and neutral equilibrium.**—Theoretically, it is always possible to balance a body by supporting it at a point anywhere either vertically *above* or vertically *below*, or *at* its C.G. But, practically, it is often very difficult to keep a body balanced on a point even for a short time, and the least disturbance either from shaking or other causes suffices to overturn it. Bodies have, as we know, a natural tendency to fall into

certain positions of equilibrium, and to fall away from other positions.

Thus an egg naturally rests on a table with its side touching the table. But it is difficult to balance the egg on its end as in Fig. 133; and, if this has been done by bringing the C.G. directly over the point of support, the egg will still overturn with the slightest shake or breath of air, or other disturbance, which moves the egg and therefore its C.G. or its point of support a little to one side or the other. In ordinary language, we express this fact by saying that the egg is *top heavy*.

Again, a crooked walking stick will readily hang down of itself with a finger supporting it underneath its crook, but the stick can only with difficulty be balanced upright on the tip of the finger, and the stick is then said to be *top heavy*.

A pin would theoretically satisfy the conditions of equilibrium of § 150 if stood upright with its point resting on a plate. But no hand is sufficiently steady and patient to place it exactly in the right position, nor could the plate and pin remain sufficiently undisturbed for the pin to continue balanced for more than an instant, even if it were so placed. The pin would in fact be *top heavy*.

On the other hand, a weight (such as a plummet) hanging from a string will of its own accord fall into a position of equilibrium with the string vertical. If pulled aside, it will at first swing to and fro, but the string will at last again assume a vertical position. Again, a round ball resting on a table can be rolled along the table, and will remain at rest if placed in any position (Fig. 119).

160. Hence there are different kinds of equilibrium, and these have received names according to the following

DEFINITIONS.—Bodies are said to be in **stable** equilibrium when they tend to return to their equilibrium position after being slightly disturbed.

Examples.—A stick hanging by its crook; a weight hanging from a string or plumbline; a ball or marble inside a basin; the beam of a balance.

Bodies are said to be in **unstable** equilibrium when the least disturbance causes them to move further and further away from their equilibrium position.

Examples.—An egg stood on end; a pin stood on its point; a walking-stick standing upright on the finger; a ball or marble placed at the top of an inverted bowl.

NOTE.—*When we say that a body is* "**top heavy,**" *we imply that it is in unstable equilibrium.*

Bodies are said to be in **neutral** equilibrium when, after being displaced, they remain in their new position.

Examples.—A ball or marble on the table; an egg resting on its side when allowed to roll along the table; a heavy body of any kind supported *at* its C.G. (§ 151); a door turning on its hinge.

A top, a hoop, or a bicycle will remain upright as long as it is in rapid motion, although it would be unstable and would overturn if at rest. Similarly the Japanese in their balancing feats are able to keep bodies standing in positions of unstable equilibrium by moving the point of support slightly, so as to counteract every tendency to overturn. Hence, when a body is *in motion*, it is impossible to argue from purely statical considerations as to whether it is stable or not.

Considerations of stability and instability of equilibrium are of the utmost practical importance, especially in shipbuilding; for, unless a ship is stable, it will capsize, as the *Victoria* did.

161. Stability of a body with one point fixed.

A heavy body, moveable freely about a fixed point O, is in stable, unstable, or neutral equilibrium, according as O is vertically above, vertically below, or at G, the body's C.G.

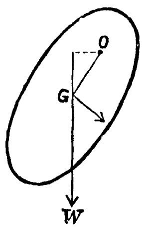

Fig. 130.

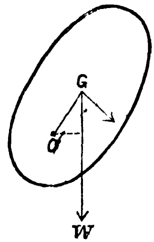

Fig. 131.

For if the line OG is not quite vertical, Figs. 130, 131 show that the moment of the weight acting at G tends to turn the body about O *towards* the position in which G is vertically below O, and *away* from the position in which G is vertically above O. Hence the former position is stable and the latter unstable. And since the body balances in every position when supported *at* G, its equilibrium is then neutral.

Thus a cake-basket can be lifted by the handles when empty, but if a large high cake is placed in it, the C.G. will be brought above the points where the basket is hinged to its handles, and it will overturn.

162. Stability of a body resting on a horizontal plane.—When a body rests touching the ground at one point, its equilibrium is not necessarily unstable, even although its C.G. is above its point of contact.

A ball whose C.G. is at its centre is in neutral equilibrium when placed on a horizontal surface. For, however the ball be rolled about, its C.G. will always be vertically above its point of contact, and hence it will always remain in equilibrium (Fig. 119, p. 158).

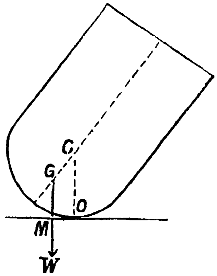

Fig. 132.

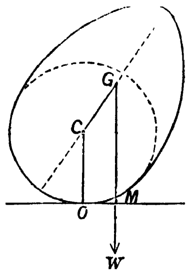

Fig. 133.

If the surface of a body is spherical and its C.G. is not at its centre, the body will be in stable equilibrium on a horizontal plane if its C.G. be vertically *below* its centre, and in unstable equilibrium if its C.G. be vertically *above* its centre; for Figs. 132, 133 show that the moment of the weight tends to turn the body *towards* the former position and *away* from the latter.

The whole body need not necessarily be spherical, provided that the part of the surface touching the plane is spherical.

Thus, if a hemisphere of lead is joined to a cylinder of cork so that the C.G. of the whole is below the centre of the hemisphere, it will stand upright although it *looks* very top-heavy. For lead is so much heavier than cork, that a cork of considerable length may be attached to the hemisphere without raising the C.G. above the centre.

Several toys act on this principle, for, if the lead and cork be coated together with paint, the difference of materials is disguised, and so a doll which will only stand on its head is easily made.

163. General condition of stability.—The weight of a body acting at its C.G. always tends to pull this C.G. down towards the Earth. Hence, if the body is in equilibrium with the C.G. in the highest possible position, its equilibrium must necessarily be unstable. And when the C.G. is at the lowest possible position, the body must necessarily be in stable equilibrium, for gravity has pulled it as low as possible, and it can go no further.

For this reason, in order to make a body stable, it is often loaded

with weights low down, so as to lower the C.G. as much as possible. This is why ships carry ballast. On the other hand, people by standing up in a small boat raise the C.G., and *may* make it top heavy.

As another illustration, a small particle or spherical ball will be in unstable equilibrium when resting at the highest point of the convex surface of a round body, and in stable equilibrium when resting at the lowest point of a concave bowl. The student should, as an exercise, draw diagrams for the two cases, showing that when the ball is displaced its weight tends to pull it away from the former but towards the latter position of equilibrium.

164. To find the C.G. of a parallelogram.

Let $ABCD$ be *either* a uniform lamina or a uniform wire in the shape of a parallelogram. Let its diagonals AC, BD intersect in G. Then shall G be the required C.G.

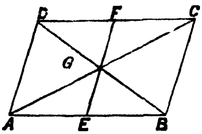

Fig. 134.

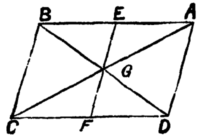

Fig. 135.

Turn the parallelogram round as in Fig. 135, and place it so that the vertices A, B, C, D coincide with the previous positions of the vertices C, D, A, B, respectively. Then the diagonal AC will coincide with its previous position CA, and BD with DB. Hence G, the intersection of the diagonals, will be unaltered in position. Also no other point in the parallelogram will occupy the same position as before. For any other point will be brought round to the opposite side of G.

Now, when the parallelogram is turned round, it occupies the same space as before; hence its C.G. in its new position coincides with its C.G. in its old position. Therefore the C.G. of the parallelogram must be at G.

OBSERVATION.—The reader should cut the parallelogram out of a sheet of paper, marking the points A, B, C, D both on the parallelogram and on the paper from which it was cut. On turning it round into the new position, it will again exactly fill the hole from which it was cut, and the diagonals will be unaltered in position, but their extremities will be interchanged. Hence G will be the intersection of the same two straight lines as before, and will therefore be unaltered in position.

COR. 1. Since the diagonals of a parallelogram bisect each other, *the* C.G. *is at the middle point of either diagonal.*

This is also obvious from the method of proof. When the parallelogram is turned round, AG is brought into coincidence with the former position of GC. Therefore $AG = GC$, and G is the middle point of AC.

COR. 2. *The* C.G. *is the middle point of the line bisecting a pair of*

For let E, F be the middle points of AB, CD. When the parallelogram is turned round, AB, CD occupy the former places of CD, AB, respectively. Hence E, F occupy the former places of F, E, respectively; and therefore the middle point of EF occupies the same position as before. Therefore this point is G, the required C.G. of the parallelogram.

COR. 3. Similarly, G is the middle point of the bisector of the pair of opposite sides BC, DA. *Hence the diagonals and the bisectors of the opposite sides of the parallelogram all bisect one another in the* C.G. *of the parallelogram,* as may be otherwise proved by geometry.

SUMMARY OF RESULTS.

The centre of gravity of a body is that point through which the resultant force due to the Earth's attraction always passes. (§ 140.)

The centre of gravity of

(1) *a straight line* is its middle point; (§ 147.)
(2) *a circular ring* is the centre of the circle; (§ 148.)
(3) *a circular area* is the centre of the circle; (§ 148.)
(4) *a parallelogram* is the intersection of its diagonals; (§ 164.)
(5) *a regular polygon* is the centre of the polygon; (§ 149.)
(6) *a sphere* is the centre of the sphere; (§ 149.)
(7) *a right cylinder* is the middle point of its axis; (§ 149.)
(8) *a cube* or *rectangular parallelepiped* is at the intersection of its diagonals. (§ 149.)

When a body is supported at one point, *its* C.G. *is in the same vertical line as the point of support.* (§ 150.)

The *base* of a body is the area enclosed by a string drawn tightly round the points of contact with the supporting surface. (§ 155.)

A body placed on a plane will *stand* or *fall* according as the vertical line through its C.G. falls *within* or *without its base.* (§ 156.)

A body is in *stable* or in *unstable equilibrium* according as when slightly displaced from its equilibrium position it tends to *return to* or *move further away from* that position. (§ 160.)

A body is in *neutral equilibrium* if when slightly displaced it remains in its new position. (§ 160.)

EXAMPLES XI.

1. Weights of 1 and 5 lbs. are fixed at the two extremities of a uniform heavy bar 3 ft. long. The centre of gravity of the whole is 1 ft. from one end. Find the weight of the bar.

2. A rod, of which the centre of mass is not at the middle point, is hung from a smooth peg by means of a string attached to its extremities. Find the positions of equilibrium.

3. A rod 12 ft. long has a weight of 1 lb. suspended from one end, and when 15 lbs. is suspended from the other end it balances at a point 3 ft. from that end, while if 8 lbs. are suspended there it balances at a point 4 ft. from that end. Find the weight of the rod and the position of its centre of gravity.

4. A rod ABC, 16 ins. long, rests in a horizontal position upon two supports at A and B 1 ft. apart, and it is found that the least upward and the least downward force applied at C which would move the rod are 4 oz. and 5 oz. respectively. Find the weight of the rod and the position of its centre of gravity.

5. Equal masses are placed at the angular points of a regular pentagon. Show that their centre of gravity is at the centre of the circumscribing circle.

6. A bar projects 6 ins. beyond the edge of a table, and when 2 oz. is hung on to the projecting end the bar just topples over; when it is pushed out so as to project 8 ins. beyond the edge 1 oz. makes it topple over. Find the weight of the bar and the distance of its centre of gravity from the end.

7. A body of any form is supported by a string, which, passing over a smooth peg, is fastened at its extremities to two points in the body. Show that in its position of equilibrium the C.G. of the body is vertically beneath the peg, and that the two portions of the string make equal angles with the vertical direction.

8. Explain fully the circumstances under which a rectangular block, standing on a plank which is being gradually tilted, shall topple over, being prevented from sliding by a small obstacle. As an example, take the case of a block 8 × 5 × 5 cubic ins.

9. Masses of 1 lb., 1 lb., 2 lbs., 2 lbs. are placed at the corners *A, B, C, D* of a rectangle *ABCD*. Determine, by the principle of § 149, a straight line in which the centres of gravity of the four masses lie.

10. Of two equal hollow spheres of the same material, one contains a quantity of water, and the other a mass of lead attached to the inner surface. Describe experiments which would enable you to distinguish between them.

11. A number of rectangular bricks are placed so that each overlaps the one next below by the greatest amount possible. Show that the distance they overlap (beginning at the top) are proportional to the fractions $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, ...

12. A crane is on a four-wheeled railway truck weighing 6 tons. The area of the *base* formed by the wheels is 50 sq. ft. Neglecting the weight of the crane, calculate the area of the ground from which a mass of 3 tons can be lifted by the crane.

CHAPTER XII.

DETERMINATION OF THE CENTRE OF GRAVITY.

165. In the last chapter we showed how the position of the C.G. of certain figures—such as the straight line, sphere, and parallelogram — can be inferred from the symmetry of the figures. In most cases, however, it is necessary to divide a body into a number of parts, and to deduce the position of the C.G. of the whole body from those of its parts. Accordingly we commence by investigating the general method applicable in such cases.

166. Having given the weights and centres of gravity of the different parts of a body, to find the C.G. of the whole body.

Let S_1, S_2, S_3 be different parts of a body, and let w_1, w_2, w_3 be the weights, K, L, M the C.G.'s, of S_1, S_2, S_3, respectively. It is required to find the C.G. of the whole body made up of the parts S_1, S_2, S_3.

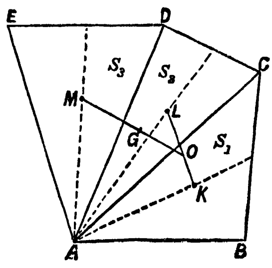

Fig. 136.

The weights w_1, w_2 may be taken to act at K, L. Therefore their resultant is a weight $w_1 + w_2$ acting at a point O on KL, such that

$$w_1 \,.\, KO = w_2 \,.\, OL.$$

$$\therefore \quad (w_1+w_2)\,KO = w_2\,.\,OL + w_2\,.\,KO = w_2\,.\,KL\,;$$

whence

$$KO = \frac{w_2}{w_1+w_2}KL, \quad OL = KL - KO = \frac{w_1}{w_1+w_2}KL.$$

Hence O is the C.G. of the body made up of S_1 and S_2.

The total weight of the two parts S_1 and S_2 is therefore w_1+w_2 acting at O.

The weight of S_3 is w_3 acting at M. Therefore their resultant is a weight $w_1+w_2+w_3$ acting at a point G, such that

$$(w_1+w_2)\,OG = w_3\,.\,GM.$$

$$\therefore \quad (w_1+w_2+w_3)\,OG = w_3\,(OG+GM) = w_3\,.\,OM,$$

whence

$$OG = \frac{w_3}{w_1+w_2+w_3}OM, \qquad GM = \frac{w_1+w_2}{w_1+w_2+w_3}OM.$$

Hence G is the C.G. of the body made up of S_1, S_2, S_3.

In like manner, if the weights and C.G.'s of four or more parts of a body are known, we can find the C.G. of the whole body. The required C.G. is evidently the C.G. of a number of particles whose weights are equal to those of the several parts of the body, placed at their respective C.G.'s, and the above method is identical with that used for finding the centre of parallel forces (p. 145).

Examples.—(1) To find the C.G. of a wire bent into an angle.

Let a uniform wire AB be bent into an angle ACB. We may consider it as consisting of two straight wires AC, CB joined together at C.

The C.G.'s of these two portions are at their middle points K, L, and their weights are proportional to their lengths AC, CB. Hence the C.G. of the whole wire is at a point G in KL, such that

$$AC \times KG = CB \times GL,$$

or

$$\frac{GK}{GL} = \frac{CB}{CA}.$$

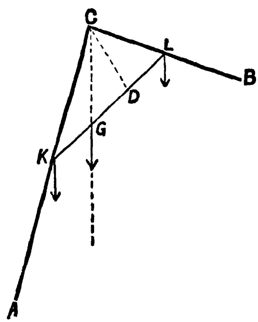

Fig. 137.

[The point G may be found by the following geometrical construction:—Draw CD bisecting the angle ACB and cutting KL in D. Take $KG = DL$, and therefore $GL = KD$. Then G is the required C.G.

For since CD bisects angle KCL, therefore (Euclid, VI. 3)

$$\frac{LD}{DK} = \frac{LC}{CK}; \qquad \therefore \quad \frac{GK}{GL} = \frac{CL}{CK} = \frac{\frac{1}{2}CB}{\frac{1}{2}CA} = \frac{CB}{CA}.\ ]$$

(2) To find the C.G. of a cubical box without a lid.

Let a be the length of a side of the cube. Let O be the centre of the cube, A the centre of its base, G the required C.G. (figure should be drawn). Then it is easy to see that the C.G. of the four sides of the box is at O, and their total area is $4a^2$. Also the C.G. of the base is at A, and its area is a^2. Therefore G is the C.G. of weights a^2 at A and $4a^2$ at O.

$$\therefore \quad a^2 \times AG = 4a^2 \times GO.$$

$$\therefore \quad AG = 4GO, \quad \text{and} \quad AO = 5GO.$$

$$\therefore \quad GO = \tfrac{1}{5}AO, \quad \text{and} \quad AG = \tfrac{4}{5}AO.$$

But $AO = \frac{1}{2}a$. Therefore $AG = \frac{2}{5}a$.

Hence the C.G. is at a height above the base of $\frac{2}{5}$ the height of the cube.

167. Definition.—The **medians** of a triangle are the straight lines joining its vertices to the middle points of its opposite sides.

Lemma. *Any straight line, parallel to the base of a triangle and terminated by its sides, is bisected by the median through the vertex opposite the base.*

Let ABC be the triangle, bc any line parallel to BC. Let D be the middle point of BC, and let AD cut bc in d. Then shall d be the middle point of bc.*

For, if not, from dc, cut off $dk = bd$. Join Ak, and join D to b, c, k.

Then, since $BD = DC$; $\therefore\ \triangle ABD = \triangle ACD$ and $\triangle bBD = \triangle cCD$,

$\therefore$ remaining $\triangle AbD = \triangle AcD$. And since $bd = dk$,

$\therefore\ \triangle Abd = \triangle Akd$, and $\triangle Dbd = \triangle Dkd$; $\therefore\ \triangle AbD = \triangle AkD$;

$\therefore\ ck$ is parallel to AD, which is impossible, since ck cuts AD in d.

* The proof which follows may be omitted and a simpler proof substituted by those who have begun to read Euclid, Book VI.

168. To find the C.G. of a triangular area.

Let ABC be a triangular lamina, D, E, F the middle points of the sides. Join AD. Then we shall show that the C.G. of the triangle lies in AD.

Suppose the triangle to be divided into a very large number of thin strips such as bc, by drawing straight lines parallel to the base BC.

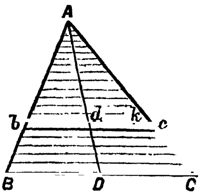

Fig. 138.

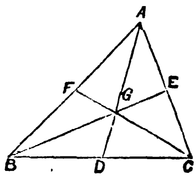

Fig. 139.

If the strips are made *sufficiently thin*, each may be treated as a uniform thin rod, and the C.G. of such a rod is at its middle point d.*

But, by the above lemma, the median AD passes through d. Hence the C.G. of every thin strip of the triangle lies in AD. And since the weight of each strip acts as if it were concentrated at its C.G., we see that the C.G. of the whole triangle is the same as that of a certain distribution of weights along the line AD.

Therefore the C.G. of the triangle lies in AD.

Similarly, by dividing the triangle up into strips parallel to AC, it may be shown that the C.G. of the triangle lies in the median line BE, and also in CF.

* We may suppose the triangle to be built up of a number of thin laths or strips of material, each slightly longer than the next above, and fixed side by side. Strictly speaking, their ends would have to be smoothed off along the sides AB, AC; but we suppose the strips so thin that the amount smoothed off is inappreciable.]

Therefore the C.G. *of the triangle is at G, the common point of intersection of the medians AD, BE, CF.*

COR. *The three medians AD, BE, CF all pass through one common point.* This is a well-known geometrical theorem, and is proved in most modern editions of Euclid.*

169. The C.G. of a triangular area coincides with the C.G. of three equal particles placed at its vertices.

Also it is the point of trisection of any median line which is the more remote from the corresponding vertex.

(i.) Let three equal weights w be placed at A, B, C.

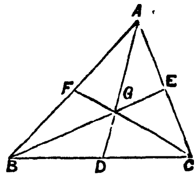

Fig. 139.

Then, if D be the middle point of BC, the weights w at B and w at C have a resultant $2w$ at D.

Hence the C.G. of the three weights is also the C.G. of weights $2w$ at D and w at A.

Therefore it lies in AD.

Similarly, it lies in BE and CF.

Therefore it is at G, the point of intersection of the three medians.

Therefore it coincides with the C.G. of the triangular area ABC

(ii.) Again, since G is the point of application of the resultant of weights $2w$ at D and w at A,

therefore G divides AD, so that

$$w \, . \, AG = 2w \, . \, GD.$$

$$\therefore \quad AG = 2GD,$$

and $$\therefore \quad AD = AG + GD = 3GD.$$

* Or see the solutions to the Geometry paper of June, 1888, in *Matriculation Model Answers.*

and $AG =$

Therefore G is that point of trisection of AD which is the more remote from the vertex A.

Cor. 1. If each of the weights w is *one-third* the weight of the lamina, their total weight and the position of their c.g. will be the same as for the lamina. Hence *a uniform triangular lamina is statically equivalent to three equal weights placed at its vertices, each being one-third the weight of the lamina.*

Cor. 2. *The point of intersection of the three medians of a triangle is one of the points of trisection of each of them.*

This may also be proved by pure geometry. The proof is left as an exercise for the reader.

170. To find the C.G. of the perimeter of a triangle.

Let ABC be a triangle. It is required to find the c.g. of a thin uniform wire bent into the triangle ABC.*

Let D, E, F be the middle points of the sides BC, CA, AB, and let their lengths be a, b, c, respectively.

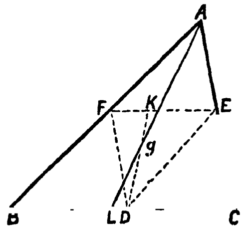

Fig. 140.

The weights of the three sides are proportional to their lengths, and act at their middle points. Hence the required c.g. is the c.g. of weights a, b, c placed at D, E, F. It might, therefore, be found by the construction of § 166.

But the following method is more convenient:—

The weight a acting at D is equivalent to weights $\frac{1}{2}a$ at B and $\frac{1}{2}a$ at C, since these have the same c.g. and the same total weight. Similarly, each of the other sides may be replaced by half its weight acting at either end. Therefore the c.g. of the perimeter is the c.g. of weights $\frac{1}{2}(b+c)$ at A, $\frac{1}{2}(c+a)$ at B, $\frac{1}{2}(a+b)$ at C.

Denoting it by g, we see from § 166 that g may be found by dividing BC in L, so that

* Note that in the present case the matter whose c.g. is required is distributed *along the sides* of the triangle, whereas in § 168 it was distributed *over the area* of the triangle.

$$(c+a)\,BL = (a+b)\,LC,$$

or $$BL = \frac{(a+b)\,a}{2a+b+c}, \qquad LC = \frac{(a+c)\,a}{2a+b+c};$$

and dividing AL in g, so that

$$Lg = \frac{b+c}{2\,(a+b+c)}\,LA.$$

[171. **Alternative construction.**—Since g is the C.G. of weights a, b, c placed at D, E, F, therefore DG produced divides EF in a point K, such that $b\,.\,EK = c\,.\,KF$. But the sides of $\triangle DEF$ are, respectively, half those of ABC.

$$\therefore \quad \frac{EK}{KF} = \frac{c}{b} = \frac{2ED}{2DF} = \frac{ED}{DF};$$

whence (Euclid, VI. 3) DK bisects $\angle EDF$. Similarly Eg, Fg bisect the angles FED, DFE.

Therefore g *is the centre of the circle inscribed in the triangle* DEF.]

172. To find the C.G. of the area of any polygon.

Divide the polygon into triangles (S_1, S_2, S_3, Fig. 141) by joining one of its vertices A to the other vertices not already joined to it. Find K, L, M, the C.G.'s of these triangles, by trisecting the medians drawn from A, as in § 169, and find their areas by measuring their bases and altitudes. Then the required C.G. of the polygon is the C.G. of weights at K, L, M proportional to the areas S_1, S_2, S_3, and is therefore given by the construction of § 166.

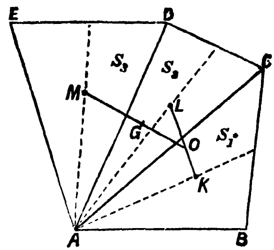

Fig. 141.

In practice, instead of replacing each triangle by a single weight acting at its C.G., it would sometimes be more convenient to replace it by three weights, each one-third of that of the triangle, placed at its angular points. We should thus reduce the problem to finding the C.G. of certain weights placed at the angular points of the polygon itself.

OBSERVATION.—The C.G. of a polygon is not, in general, the C.G. of *equal weights* placed at all the corners of the polygon, for, if the last construction be followed, the weights by which different triangles are replaced are not, in general, equal.

Example.—To find the C.G. *of a trapezoid* (*i.e.*, a quadrilateral two of whose sides are parallel).

Let $ABDC$ be the trapezoid, having AC, BD parallel. Let $AC = a$, $BD = b$, and let h be the perpendicular distance between AC and BD.

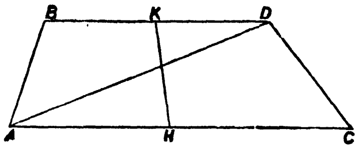

Fig. 142.

The quadrilateral may be divided into two triangles ADC, ABD, and the areas of these triangles are $\frac{1}{2}ha$, $\frac{1}{2}hb$; therefore their weights are proportional to a, b.

Their C.G. will therefore be unaltered by supposing their weights to be *equal* to a, b, respectively.

Then the C.G. and weight of the triangle ADC will be the same as that of equal weights $\frac{1}{3}a$ placed at A, D, C. Similarly, the triangle ABD may be replaced by equal weights $\frac{1}{3}b$ at A, B, D.

Hence the C.G. of the quadrilateral is the C.G. of the weights

$$\tfrac{1}{3}(a+b) \text{ at } A, \quad \tfrac{1}{3}a \text{ at } C, \quad \tfrac{1}{3}b \text{ at } B, \quad \text{and} \quad \tfrac{1}{3}(a+b) \text{ at } D.$$

Again, by dividing the quadrilateral into two triangles ABC, BDC, we see that its C.G. is also the C.G. of weights

$$\tfrac{1}{3}a \text{ at } A, \quad \tfrac{1}{3}(a+b) \text{ at } C, \quad \tfrac{1}{3}(a+b) \text{ at } B, \quad \text{and} \quad \tfrac{1}{3}b \text{ at } D.$$

Hence the C.G. will be the same when the latter weights are added to the former, and it is therefore the C.G. of weights

$$\tfrac{1}{3}(2a+b) \text{ at } A, \quad \tfrac{1}{3}(2a+b) \text{ at } C, \quad \tfrac{1}{3}(a+2b) \text{ at } B, \quad \tfrac{1}{3}(a+2b) \text{ at } D.$$

Let H, K be the middle points of AC, BD.

Then the weights $\frac{1}{3}(2a+b)$ at A and $\frac{1}{3}(2a+b)$ at C are equivalent to a weight $\frac{2}{3}(2a+b)$ at H.

Similarly, the weights $\frac{1}{3}(a+2b)$ at B and $\frac{1}{3}(a+2b)$ at D are equivalent to a weight $\frac{2}{3}(a+2b)$ at K.

Hence the C.G. of the quadrilateral is in HK at a point G, such that

$$\tfrac{2}{3}(2a+b) \times HG = \tfrac{2}{3}(a+2b) \times GK,$$

or

$$\frac{HG}{GK} = \frac{a+2b}{2a+b}.$$

173. To find the C.G. of a portion of a body.

Having given the weight and C.G. of a whole body and of any part removed from it, to find the position of the C.G. of the remaining part.

Let O be the C.G. of a body, W its weight.

Let C be the C.G. of any part of the body, w its weight.

Let G be the C.G. of the remainder of the body. It is required to find G.

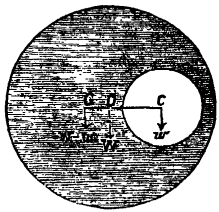

Fig. 143.

The weight of this remaining part is evidently $W-w$.

Now, since O is the C.G. of the whole body, O is the C.G. of weights w at C and $W-w$ at G.

$$\therefore \quad C, O, G \text{ lie in a straight line, and}$$

$$(W-w)\,GO = w \,.\, OC.$$

Therefore G lies on CO produced through O, so that

$$OG = \frac{w}{W-w}\,CO.$$

Example.—To find the C.G. of a hollow spherical bullet containing an excentric spherical cavity.

Let O be the centre of the surface of the bullet, C the centre of the cavity.

Let a, b be the known radii of the bullet and cavity, and let $OC = c$. Then the volumes of the bullet and of the cavity are, respectively, $\frac{4}{3}\pi a^3$ and $\frac{4}{3}\pi b^3$.

Hence, if W denote the weight which the bullet would have if no cavity existed, w the weight of matter which would fill the cavity,

$$\frac{w}{W} = \frac{b^3}{a^3},$$

and the weight of the actual bullet is $W-w$.

Hence the required C.G. of the hollow bullet is a point G in CO produced, such that

$$OG = CO \times \frac{w}{W-w},$$

or

$$OG = c \times \frac{b^3}{a^3-b^3}.$$

174. To find the C.G. of any number of weights at given points in one plane.

Let any number of particles of known weights w_1, w_2, w_3 be situated at given points A_1, A_2, A_3 in one plane. Draw two straight lines OX, OY at right angles to one another in the plane, and let the distances of each weight from each of these two lines be measured.*

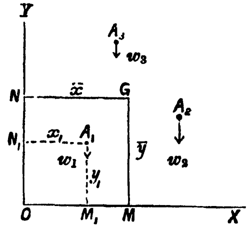

Fig. 144.

Let x_1, x_2, x_3, ... be the distances of the weights from OY; y_1, y_2, y_3, ... their distances from OX.

[So that if, from any weight A_1, perpendiculars A_1M_1 on OX and A_1N_1 on OY be drawn, we have

$$x_1 = OM_1 = N_1A_1 \quad \text{and} \quad y_1 = ON_1 = M_1A_1.]$$

Let G be the required C.G. of the weights. Draw GM, GN perpendicular on OX, OY, and let $\bar{x} = OM = NG$, $\bar{y} = ON = MG$.

The resultant of the weights w_1, w_2, w_3, &c., acting at A_1, A_2, A_3, is weight of $w_1 + w_2 + w_3 \ldots$ acting at G.

Suppose the plane turned so that OY is vertical and OX horizontal. Then, since the sum of the moments of the several weights about O is equal to the moment of their resultant,

$$\therefore \quad OM \times (w_1 + w_2 + w_3 + \ldots) = OM_1 \times w_1 + OM_2 \times w_2 + OM_3 \times w_3 + \ldots,$$

or
$$\bar{x} = OM = \frac{w_1x_1 + w_2x_2 + w_3x_3 + \ldots}{w_1 + w_2 + w_3 + \ldots}.$$

Next place the system so that OX is vertical and OY horizontal.

By taking moments in like manner about O, we have

$$ON \times (w_1 + w_2 + w_3 + \ldots) = ON_1 \times w_1 + ON_2 \times w_2 + ON_3 \times w_3 + \ldots,$$

whence
$$\bar{y} = ON = \frac{w_1y_1 + w_2y_2 + w_3y_3 + \ldots}{w_1 + w_2 + w_3 + \ldots}.$$

[* We may suppose the "particles in one plane" to be a number of small weights attached to a flat square sheet of cardboard, and the two straight lines at right angles to be two adjacent edges of the square, or instead of the paper we may take a school slate, when two adjacent sides of the frame will represent OX, OY, and the position of the particles may be represented on the slate.]

173. To find the C.G. of a portion of a body.

Having given the weight and C.G. of a whole body and of any part removed from it, to find the position of the C.G. of the remaining part.

Let O be the C.G. of a body, W its weight.

Let C be the C.G. of any part of the body, w its weight.

Let G be the C.G. of the remainder of the body. It is required to find G.

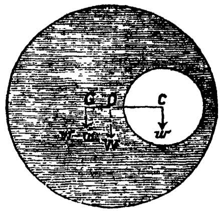

Fig. 143.

The weight of this remaining part is evidently $W-w$.

Now, since O is the C.G. of the whole body, O is the C.G. of weights w at C and $W-w$ at G.

$$\therefore \quad C,\ O,\ G \text{ lie in a straight line, and}$$
$$(W-w)\,GO = w\,.\,OC.$$

Therefore G lies on CO produced through O, so that

$$OG = \frac{w}{W-w}\,CO.$$

Example.—To find the C.G. of a hollow spherical bullet containing an excentric spherical cavity.

Let O be the centre of the surface of the bullet, C the centre of the cavity.

Let a, b be the known radii of the bullet and cavity, and let $OC = c$. Then the volumes of the bullet and of the cavity are, respectively, $\frac{4}{3}\pi a^3$ and $\frac{4}{3}\pi b^3$.

Hence, if W denote the weight which the bullet would have if no cavity existed, w the weight of matter which would fill the cavity,

$$\frac{w}{W} = \frac{b^3}{a^3},$$

and the weight of the actual bullet is $W-w$.

Hence the required C.G. of the hollow bullet is a point G in CO produced, such that $$OG = CO \times \frac{w}{W-w},$$

or $$OG = c \times \frac{b^3}{a^3-b^3}.$$

173. To find the C.G. of a portion of a body.

Having given the weight and C.G. of a whole body and of any part removed from it, to find the position of the C.G. of the remaining part.

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Let C be the C.G. of any part of the body, w its weight.

Let G be the C.G. of the remainder of the body. It is required to find G.

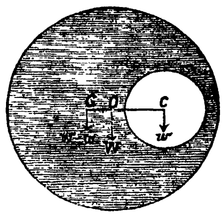

Fig. 143.

The weight of this remaining part is evidently $W-w$.

Now, since O is the C.G. of the whole body, O is the C.G. of weights w at C and $W-w$ at G.

$\therefore$ C, O, G lie in a straight line, and

$$(W-w)\,GO = w\,.\,OC.$$

Therefore G lies on CO produced through O, so that

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Let a, b be the known radii of the bullet and cavity, and let $OC = c$. Then the volumes of the bullet and of the cavity are, respectively, $\frac{4}{3}\pi a^3$ and $\frac{4}{3}\pi b^3$.

Hence, if W denote the weight which the bullet would have if no cavity existed, w the weight of matter which would fill the cavity,

$$\frac{w}{W} = \frac{b^3}{a^3},$$

and the weight of the actual bullet is $W-w$.

Hence the required C.G. of the hollow bullet is a point G in CO produced, such that $$OG = CO \times \frac{w}{W-w},$$

or $$OG = c \times \frac{b^3}{a^3-b^3}.$$

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Having given the weight and C.G. of a whole body and of any part removed from it, to find the position of the C.G. of the remaining part.

Let O be the C.G. of a body, W its weight.

Let C be the C.G. of any part of the body, w its weight.

Let G be the C.G. of the remainder of the body. It is required to find G.

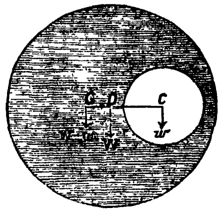

Fig. 143.

The weight of this remaining part is evidently $W-w$.

Now, since O is the C.G. of the whole body, O is the C.G. of weights w at C and $W-w$ at G.

$\therefore$ C, O, G lie in a straight line, and

$$(W-w)\,GO = w\,.\,OC.$$

Therefore G lies on CO produced through O, so that

$$OG = \frac{w}{W-w}\,CO.$$

Example.—To find the C.G. of a hollow spherical bullet containing an excentric spherical cavity.

Let O be the centre of the surface of the bullet, C the centre of the cavity.

Let a, b be the known radii of the bullet and cavity, and let $OC = c$. Then the volumes of the bullet and of the cavity are, respectively, $\frac{4}{3}\pi a^3$ and $\frac{4}{3}\pi b^3$.

Hence, if W denote the weight which the bullet would have if no cavity existed, w the weight of matter which would fill the cavity,

$$\frac{w}{W} = \frac{b^3}{a^3},$$

and the weight of the actual bullet is $W-w$.

Hence the required C.G. of the hollow bullet is a point G in CO produced, such that

$$OG = CO \times \frac{w}{W-w},$$

or

$$OG = c \times \frac{b^3}{a^3-b^3}.$$

174. To find the C.G. of any number of weights at given points in one plane.

Let any number of particles of known weights w_1, w_2, w_3 be situated at given points A_1, A_2, A_3 in one plane. Draw two straight lines OX, OY at right angles to one another in the plane, and let the distances of each weight from each of these two lines be measured.*

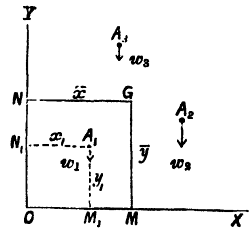

Fig. 144.

Let x_1, x_2, x_3, ... be the distances of the weights from OY; y_1, y_2, y_3, ... their distances from OX.

[So that if, from any weight A_1, perpendiculars A_1M_1 on OX and A_1N_1 on OY be drawn, we have

$$x_1 = OM_1 = N_1A_1 \quad \text{and} \quad y_1 = ON_1 = M_1A_1.]$$

Let G be the required C.G. of the weights. Draw GM, GN perpendicular on OX, OY, and let $\bar{x} = OM = NG$, $\bar{y} = ON = MG$.

The resultant of the weights w_1, w_2, w_3, &c., acting at A_1, A_2, A_3, is weight of $w_1 + w_2 + w_3$ ... acting at G.

Suppose the plane turned so that OY is vertical and OX horizontal. Then, since the sum of the moments of the several weights about O is equal to the moment of their resultant,

$$\therefore \quad OM \times (w_1 + w_2 + w_3 + \ldots) = OM_1 \times w_1 + OM_2 \times w_2 + OM_3 \times w_3 + \ldots,$$

or
$$\bar{x} = OM = \frac{w_1x_1 + w_2x_2 + w_3x_3 + \ldots}{w_1 + w_2 + w_3 + \ldots}.$$

Next place the system so that OX is vertical and OY horizontal.

By taking moments in like manner about O, we have

$$ON \times (w_1 + w_2 + w_3 + \ldots) = ON_1 \times w_1 + ON_2 \times w_2 + ON_3 \times w_3 + \ldots,$$

whence
$$\bar{y} = ON = \frac{w_1y_1 + w_2y_2 + w_3y_3 + \ldots}{w_1 + w_2 + w_3 + \ldots}.$$

[* We may suppose the "particles in one plane" to be a number of small weights attached to a flat square sheet of cardboard, and the two straight lines at right angles to be two adjacent edges of the square, or instead of the paper we may take a school slate, when two adjacent sides of the frame will represent OX, OY, and the position of the particles may be represented on the slate.]

Hence the distances OM, ON are known, and by completing the rectangle $OMGN$, the position of G, the required C.G., can be found.

OBSERVATIONS.—The formulæ evidently apply to finding the C.G. of a body, when A_1, A_2, A_3 are the C.G.'s of its several portions, and w_1, w_2, w_3 their weights.

Again, if for w_1, w_2, w_3 we write m_1, m_2, m_3, the masses of the bodies, the formulæ determine the position of their centre of mass.

Although the weights are supposed above to be in one plane, we may here state that similar formulæ hold in the more general case. If x_1, y_1, z_1 are the distances of any weight A_1 from three planes at right angles (say two adjacent walls and the floor of a room), and so on, the corresponding distances $\bar{x}$, $\bar{y}$, $\bar{z}$ for the C.G. are given by formulæ proved above, and a third similar formula with z's written for x's or y's.

Example.—(1) To find the C.G. of a square slate $ABCD$ whose weight is 1 lb., together with weights of 2, 3, 4, 5 lbs., placed at its four corners.

Let x be the distance of the C.G. from AD, y its distance from AB, a the length of the side of the square.

The total weight $= 1+2+3+4+5$ lbs. $= 15$ lbs. Therefore, taking AB horizontal, the equation of moments about A gives

$$15x = 2 \cdot 0 + 5 \cdot 0 + 1 \cdot \tfrac{1}{2}a + 3 \cdot a + 4 \cdot a = 7\tfrac{1}{2}a;$$

$$\therefore \quad x = \tfrac{1}{2}a.$$

Taking AD horizontal, we have, in like manner,

$$15y = 2 \cdot 0 + 3 \cdot 0 + 1 \cdot \tfrac{1}{2}a + 4 \cdot a + 5 \cdot a = 9\tfrac{1}{2}a;$$

$$\therefore \quad y = \tfrac{19}{30}a.$$

Hence G lies on the bisector of the sides AB, DC at a distance $\frac{19}{30}a$ from AB.

175. Work done in raising weights. — *The work done in raising a number of weights off the ground or raising them up to the ground is the same as if their total weight were collected at their* C.G.

Let there be any number of weights w_1, w_1 w_3, ..., and let them be raised from the ground to heights x_1, x_2, x_3, ... Let W be their total weight, $\bar{x}$ the height of their C.G. in the new position.

Then work done in raising the weights

$$= w_1x_1 + w_2x_2 + w_3x_3 + \ldots.$$

But, by the last article,

$$\bar{x} = \frac{w_1x_1 + w_2x_2 + w_3x_3 + \ldots}{w_1 + w_1 + w_3 + \ldots} = \frac{w_1x_1 + w_2x_2 + w_3x_3 + \ldots}{W};$$

$$\therefore \quad w_1x_1 + w_2x_2 + \ldots = W \cdot \bar{x};$$

$\therefore$ whole work done $= W\bar{x} =$ work required to lift total weight to the height of the C.G.

Similarly, by taking $x_1, x_2 \dots$ to represent the depths of a number of weights *below* the ground, we see that the work done in lifting a number of weights from below the surface to the ground is the same as if the weights were all concentrated at their C.G.

OBSERVATION.—The weights need not all be in the same vertical plane. *The theorem is true in every case*, but, unless the weights are in a vertical plane, the above proof assumes the more general theorem which we stated without proof in the observation on § 174.

Examples.—(1) The work done in building a cylindrical tower is the same as would be required to lift the whole of the materials through $\frac{1}{2}$ the height of the tower.

(2) The work done in digging a ditch of triangular section through earth of uniform material is the same as would be required to lift the total mass of earth through $\frac{1}{3}$ the depth of the lowest point of the ditch.

176. To find the C.G. of a uniform *tetrahedron* or pyramid on a triangular base.—We shall now show* that—

(i.) *The* C.G. *of a triangular pyramid $ABCD$ is in the line joining any corner D to H, the* C.G. *of the opposite face.*

(ii.) *It coincides with the* C.G. *of four equal weights placed at the corners A, B, C, D.*

(iii.) *It divides the straight line joining any corner to the* C.G. *of the opposite face in the ratio of* 3 *to* 1, *i.e., at a point G, such that $DG = \frac{3}{4}DH$ and $GH = \frac{1}{4}DH$.*

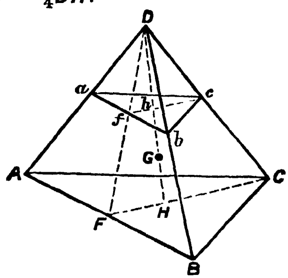

Fig. 145.

(i.) Let F be the middle point of AB, H the C.G. of the triangle ABC. It is required to show that the C.G. of the pyramid lies in DH.

Divide the pyramid into an infinitely large number of infinitely thin triangular laminæ of uniform thickness by drawing planes parallel to the face ABC, and let abc be one of these sections.

Join DF, cutting ab in f, and in the plane of the triangle DCF draw the straight lines cf, DH intersecting in h.

* The proofs may be omitted by the beginner, but the results stated are important.

Then, since ab is parallel to AB and F is the middle point of AB, therefore f is the middle point of ab (§ 167). Hence ch produced bisects ab in f; therefore ch is a median of the triangle abc. In like manner, ah and bh are medians of abc. Therefore h is the C.G. of the triangular lamina abc, and it lies in the line DH. Thus the C.G. of each of the laminæ of the pyramid is in the line DH; therefore that of the pyramid is also in the line DH.

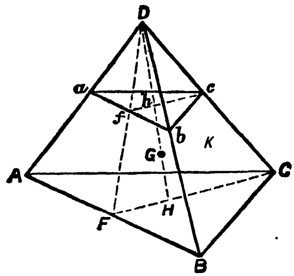

Fig. 145.

(ii.) Now let equal weights w be placed at the four corners A, B, C, D. The C.G. of the weights at A, B, C is H; hence that of the four weights also lies in the line DH.

Similarly, if either of the other corners, such as A, be joined to K, the C.G. of the opposite face, both the C.G. of the pyramid and that of of the four equal weights at A, B, C, D must lie in the joining line AK.

Therefore AK and DH must intersect in a point G, and G will be the C.G. both of the pyramid and of the four equal weights at A, B, C, D.

(iii.) Now the C.G. of the four weights w at A, B, C, D is the same as that of w at D, and $3w$ at H (since H is the C.G. of the weights at A, B, C). Therefore G, the C.G. of the pyramid, is a point in DH, such that

$$3GH = DG, \quad \text{or} \quad DG : GH = 3 : 1,$$

whence $$GH = \tfrac{1}{4}DH, \quad DG = \tfrac{3}{4}DH.$$

COR. 1. G is the middle point of the line joining the middle points of opposite edges of the pyramid.

This follows at once by replacing the weights w at A, B by a single weight $2w$ at F, the middle point of AB, and the weights w at C, D by $2w$ at the middle point of CD.

COR. 2. The lines joining the four corners of a triangular pyramid to the C.G.'s of the opposite faces all pass through one common point G, and are there divided in the ratio of 3 to 1. Also the lines joining the middle points of pairs of opposite edges pass through and are bisected at the same point.

177. To find the C.G. of a pyramid on any polygonal base whatever.

Let V be the vertex, $ABCDE$ the polygonal base of the pyramid. Let H be the C.G. of the base. Then shall the C.G. of the pyramid be a point G on VH, such that

$$VG = 3GH, \quad \text{or} \quad GH = \tfrac{1}{4}VH;$$

and $\therefore$ $VG = \tfrac{3}{4}VH.$

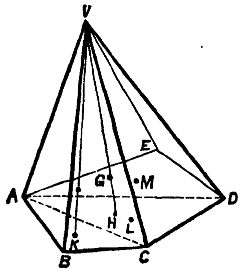

Fig. 146.

For divide the pyramid into a number of triangular pyramids, having V for vertex and the triangles ABC, ACD, ADE for bases. Let K, L, M be the C.G.'s of these bases.

Consider the triangular pyramid $VABC$. Its C.G. divides VK in the proportion of $3 : 1$; therefore it may be replaced by $\frac{1}{4}$ its weight at V and $\frac{3}{4}$ its weight at K.

Let the weights of the other triangular pyramids be similarly replaced. Then the weights thus placed at K, L, M are proportional to the volumes of the pyramids, and therefore to the areas of their bases (since they have the same altitude). Hence the C.G. of these weights is H, the C.G. of the area of the base.

Therefore the whole pyramid is replaced by $\frac{1}{4}$ its total weight at V and $\frac{3}{4}$ its total weight at H. Therefore its C.G. divides VH so that $VG = 3GH$, as was to be proved.

178. To find the C.G. of a cone.

Draw any polygon circumscribing the base of the cone and complete the pyramid, having this polygon for base, and having its vertex at the vertex of the cone. This pyramid will circumscribe the cone, but, if the number of faces of the pyramid be made sufficiently great, the pyramid will not differ perceptibly from the cone.*

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* For fresh faces may be added by slicing off the sharp edges of the slant surface of the pyramid, and this process continued till the pyramid is smoothed down to a cone.

Hence what holds good for the C.G. of a pyramid must also hold good for the C.G. of a cone.

Therefore the C.G. of a cone is in the line joining the vertex to the C.G. of the base, and at a distance from the latter point equal to $\frac{1}{4}$ the distance of the vertex.

In a *right circular cone* (the only kind of cone we have to consider) the C.G. is, of course, in the *axis*, at a distance from the base of $\frac{1}{4}$ the altitude of the cone.

179. **To find the C.G. of the *slant surface* of a regular pyramid, and of a right circular cone.**

The slant surface of the pyramid whose vertex is V and base any polygon $ABCDE$... consists of a number of triangles VAB, VBC, VCD. The C.G. of each triangle is at a distance from its base equal to $\frac{1}{3}$ the distance of the vertex V.

Hence the C.G. of the whole surface is at a height above the base equal to $\frac{1}{3}$ the altitude of the pyramid.

And therefore, if the pyramid is *regular*, the C.G. must lie in its axis VH (which is perpendicular to and passes through H the centre of the base) at a point G, such that

$$HG = \tfrac{1}{3}HV \quad \text{and} \quad GV = \tfrac{2}{3}HV.$$

If the number of faces of the pyramid be sufficiently increased, the slant surface of the pyramid will ultimately become the slant surface of a cone.

Hence the C.G. of the slant surface of a right circular cone is on its axis, at a distance from the base equal to $\frac{1}{3}$ of the altitude.

Observations.—In the above investigation, the surface of the *base* is not taken into account. If we wanted the C.G. of the whole surface-area, including the base, we should have to apply § 166.

The C.G. of the slant surface of *any* pyramid or cone whatever is at a distance from the base equal to $\frac{1}{3}$ the altitude; but, unless the pyramid or cone is symmetrical about its axis, there is no simple rule for finding the point where the line VG produced meets the base. This point is *not*, in general, the C.G. of either the area or perimeter of the base.

Summary of Results.

The centre of gravity of

(1) a *triangular area* is the point of trisection of any median furthest from the corresponding vertex. It coincides with the C.G. of three equal particles placed at its vertices. (§§ 168, 169.)

(2) *a pyramid* or *a right cone* is in the line joining its vertex to the C.G. of the base at a distance from the base equal to $\frac{1}{4}$ of the distance of the vertex from the base. (§§ 176–178.)

If W, w be the weights of a body and of a portion of it, and if their C.G.'s are at O and C, the C.G. of the remaining portion is at a point G in CO produced through O, such that

$$OG = \frac{w}{W-w} CO. \qquad (\S\ 173.)$$

If $x_1, x_2, x_3 \ldots, y_1, y_2, y_3 \ldots$ be the distances of a series of weights $w_1, w_2, w_3 \ldots$ from two perpendicular straight lines OY, OX, respectively, the distances $\bar{x}$, $\bar{y}$ of the C.G. of these weights from OY, OX are given by

$$\bar{x} = \frac{w_1 x_1 + w_2 x_2 + \ldots}{w_1 + w_2 + \ldots}, \quad \bar{y} = \frac{w_1 y_1 + w_2 y_2 + \ldots}{w_1 + w_2 + \ldots}. \qquad (\S\ 174.)$$

The *work done in raising a number of weights* is the same as if their total weight were collected at their C.G. (§ 175.)

EXAMPLES XII.

1. How would you determine the centre of gravity of a hoop that was not quite circular?

2. A uniform isosceles triangle has its two equal sides each 5 ft. long, and its base 8 ft. long; find its centre of gravity. If its weight be 5 lbs., and a weight of 10 lbs. be hung at the vertex, find the centre of gravity of the whole.

3. Weights of 2 lbs., 7 lbs., 9 lbs., 4 lbs. are placed at the corners A, B, C, D, respectively, of a square, the length of whose side is 1 ft. Find the position of the C.G. of the weights.

4. Weights of 3 lbs., 5 lbs., 2 lbs., and 6 lbs. are placed at the corners A, B, C, D, respectively, of a square, the length of whose side is 8 ins. Find the position of the centre of gravity of the weights.

5. $ABCD$ is a square, O the point of intersection of its diagonals, E, F the middle points of the sides AB, AD. If the square $AEOF$ be removed, find the centre of gravity of the remainder.

6. Find the centre of gravity of the remaining portion of a parallelo-

gram when a triangle has been cut off from the parallelogram by a single straight cut.

7. *ABCD* is a square; *E* and *F* are the middle points of *AB, AD.* If the triangle *AEF* be removed, find the centre of gravity of the remaining area.

8. Assuming the rule for finding the centre of gravity of a triangular pyramid, prove the rule for finding that of a pyramid whose base is a four-sided figure.

9. Prove that if equal triangles be cut from the corners of a given triangle by lines parallel to the respective opposite sides the centre of mass of the remainder will coincide with that of the triangle.

10. If there are two triangles on the same base and between the same parallels, prove that the distance between their centres of gravity is one-third of the distance between their vertices.

11. *ABCD* is a square, *O* the intersection of its diagonals. If the triangle *AOB* is removed, find the centre of gravity of the remainder.

12. If the angular points of one triangle lie at the middle points of the sides of another, show that the centres of gravity of the triangles are coincident.

13. A cylinder of metal is 1 ft. high, 1 ft. external diameter and 11 ins. internal diameter and 11 ins. deep inside. It is open at the top. Find the position of its centre of gravity.

14. Determine the C.G. of a cone and of a frustum of a cone.

15. A corner of a square sheet of paper is doubled over to the middle of the square. Show that the centre of gravity of the paper is at a distance from the centre equal to $\frac{1}{48}$ of the diagonal.

16. Find the number of foot-pounds of work required to wind up a given chain which hangs by one end.

17. Find the C.G. of a circular board, from which a circular piece has been cut out, having as diameter a radius of the board.

18. Prove that a triangular plate cannot stand vertically with its base resting on a horizontal plane if its vertex overhangs the base by more than the length of the base.

19. Three particles are situate at A, B, C. Prove that, by properly adjusting the ratios of their masses, their centre of gravity can be made to occupy any given position within the triangle ABC.

20. Find the ratios of the masses in Question 19 when the centre of gravity is at the centre of the inscribed circle.

21. A circular wire is divided into two parts by a chord. Prove that the distances of the centres of gravity of the parts from the centre of the circle are inversely proportional to the lengths of the parts.

22. A solid figure is formed of an upright triangular prism surmounted by a pyramid; if the length of every edge of this figure be a ft., find the height of its centre of gravity above the base.

23. A square uniform plate is suspended at one of its vertices, and a weight equal to half that of the plate is suspended from an adjacent vertex of the square. Find the point where the vertical through the point of suspension cuts the opposite diagonal of the square.

24. Two sides of a rectangle are double of the other two, and on the longer side an equilateral triangle is described. Find the centre of gravity of the lamina made up of the rectangle and the triangle.

25. Masses of 1, 2, 3, 4, 5, 6, 7, 8 lbs. respectively are placed at the corners of a cube $ABCDEFGH$, whose edge is 1 ft. and whose faces $ABCD$, $EFGH$ are horizontal, $ABCD$ being uppermost. How many ft.-lbs. of work are done in exchanging the masses at A, B, C, D with those at E, F, G, H? Hence find the vertical distance through which the common centre of gravity has been raised.

26. A shaft, 560 ft. deep and 5 ft. in diameter, is full of water; how many ft.-lbs. of work are required to empty it?

27. ABC is a plane triangle. Weights of 2 lbs., 2 lbs., and 1 lb. are placed at the vertices, and their centre of gravity G is found. Then weights of 8 oz., 8 oz., and 14 oz. are placed at the same vertices, and their centre of gravity H is found. Prove that G and H are equally distant from the centre of gravity of the triangle.

28. Two circles of radii a, b touch one another internally, and the space between them is cut out of paper. Show that the distance of its C.G. from the point of contact of the circles is $(a^2+ab+b^2) \div (a+b)$, and find this distance when b approaches and becomes equal to a.

EXAMINATION PAPER V.

1. Define the centre of a system of parallel forces, and state a method of finding its position when there are more than two forces.

2. If a body be suspended from a point, prove that its centre of gravity is vertically below that point.

3. Having given the C.G. of a body and that of one part, find the C.G. of the remainder.

4. Prove that the centre of gravity of a uniform triangular area coincides with that of three equal heavy particles placed at its angular points.

5. The centre of gravity of a quadrilateral lies in one of the diagonals. Prove that one of the diagonals is bisected by the other.

6. How would you test the nature of the equilibrium of a body at rest?

Point out the advantages of three-legged and four-legged tables respectively.

7. Find the centre of six like parallel forces of 3 lbs., 2 lbs., 3 lbs., 4 lbs., 5 lbs., 2 lbs. acting at points A, B, C, D, E, F in a straight line such that AB, BC, CD, DE, EF equal 2, 2, 4, 4, and 3 ins. respectively.

Where would the resultant act if the forces at D and E were in the opposite direction to the rest?

8. The lid of a cubical box of uniform thickness is turned back till it comes into a horizontal position. If each edge of the box measures 12 ins., what is the position of the centre of gravity of the box?

9. Find the centre of gravity of a uniform triangular pyramid; and prove that its position coincides with that of four spheres of equal weight centred at the four angles of the pyramid.

10. The triangle formed by joining the middle points of the sides of a given triangle is removed. Find the position of the centre of gravity of the remainder.

CHAPTER XIII.

BALANCES.

180. In this chapter we shall describe the various contrivances by which bodies are usually **weighed.** Remembering that weight is proportional to mass, we observe that the operation of weighing by balancing a body with known weights affords in every case a correct measure of the *mass* or quantity of matter in the body in pounds or grammes or other chosen units, and that the observed *weight* is independent of any local variations in the intensity of gravity (*Dynamics*, Chap. VIII.).

181. The common balance (see Fig. 147, p. 192) consists essentially of a **beam** or lever AB fixed so that it can turn about a fulcrum O placed a little above its middle point. From its ends are suspended two scale pans; the goods to be weighed are placed in one of these, and are balanced by placing suitable weights in the other, till the beam assumes a horizontal position.

In delicately constructed balances, the fulcrum and points of suspension consist of wedge-shaped pieces of hard steel (called "**knife blades**"), whose edges rest on hard plates of steel.

The requisites of a good balance are that it be

(i.) **true,** (ii.) **stable,** (iii.) **sensitive,** (iv.) **rigid.**

182. **Conditions to be satisfied by a true balance.**—A balance is said to be **true** if the beam assumes a horizontal position when equal weights are placed in the two scale pans. This requires that—

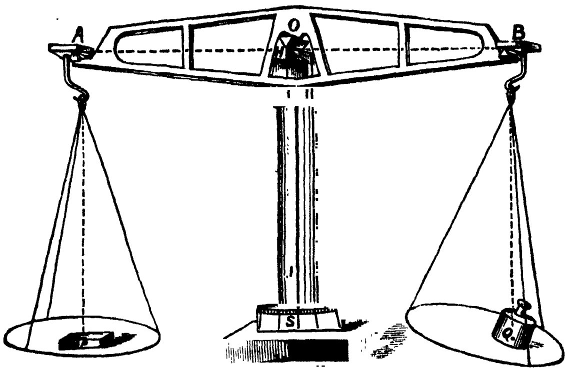

Fig. 147.

(i.) *The two arms of the beam must be of equal length*, that is, $AO = BO$, or the fulcrum O must be in a line HO, bisecting at right angles the line AB, which joins the points of suspension of the two scale pans.

(ii.) *The scale pans must be of equal weight.*

(iii.) *The* C.G. *of the beam must be vertically under the fulcrum when the beam is horizontal*, and therefore also in HO.

When these conditions are satisfied, equal weights placed *anywhere whatever* in the scale pans will balance each other with the beam horizontal.

For, since the scale pans hang freely from the beam at A and B, they will assume positions in which the C.G. of each pan and its contents is

vertically below its points of suspension.* Hence the total weights of the pans and their contents always act vertically through *A*, *B* (Fig. 148, p. 194). Those weights are equal; therefore their resultant acts at *H*, the middle point of *AB*. Also the weight of the beam acts at *G*, its C.G. But when the beam is horizontal, *G*, *H* are both vertically below the fulcrum *O*. Hence the resultant forces at *G*, *H* have no moment about *O*, and the beam is in equilibrium.

183. **Conditions that the balance may be stable.**—A balance is said to be **stable** if the beam tends of its own accord to fall into its equilibrium position. A balance would evidently be useless for weighing if its equilibrium position were *unstable* or even *neutral* (§ 160).

A balance is said to be more or less stable according to the comparative readiness or reluctance of the beam to assume its equilibrium position

Stability is secured by placing *O*, the fulcrum of the beam, a little *above* the points *G*, *H*, at which the resultant weights of the beam and the two pans act respectively.

For, if the beam be slightly inclined (Fig. 148), the equal weights of the two loaded scale pans still act at *A*, *B*, and their resultant therefore still acts at *H*†. And the moments about *O* of this resultant at *H*, and the weight of the beam at *G*, both tend to restore equilibrium by bringing the beam back to its horizontal position.

If the fulcrum *O* is only at a *very little* height above *G*, *H*, the moment tending to restore equilibrium will be very small, and the balance will oscillate for a long time before coming to rest, and will therefore possess very little stability. If goods have to be weighed quickly, the stability may be increased by increasing the height of the fulcrum *O*, thereby increasing the moment about *O* for any inclination of the beam. This, however, diminishes the sensitiveness of the balance, as we shall now explain.

* If one of the weights (*Q*, Fig. 147) is placed a little on one side in the scale pan instead of in the middle, the pan will swing itself a little to the other side so as to bring its C.G. vertically below the point of support *B*. The student should verify this by experiment.

† The point *H* is not the C.G. of the scale pans and their contents, but is the centre of parallel forces for their weights acting at *A*, *B*. It may or may not coincide with *G*.

184. Conditions that the balance may be sensitive.—It is not sufficient that a balance should show when the weights in the scale pans are equal. It must also indicate when they are unequal by the beam assuming a non-horizontal position. This is expressed by saying that the balance must be **sensitive** (or, as some writers call it, "sensible"). In a sensitive balance, a small additional weight placed in one scale pan should turn the beam through a perceptible angle, and the smallest weight which suffices to do this affords a measure of the sensitiveness and of the degree of accuracy attainable in weighing.

Thus a good chemical balance will indicate differences of weight down to tenths of a milligramme.

In order to enable the smallest deflection to be observed with great accuracy, the beam carries a vertical index or pointer I, which moves in front of a fixed graduated scale S.

Sensitiveness may be secured at the expense of stability by reducing the height of the fulcrum O, and by lengthening the arms.

For let a be the length of the arms AH, HB, Q the weight of each scale-pan and its contents, W that of the beam.

Let a small additional weight w be placed in the scale-pan at A. The moment of this weight about O is initially wa, and it turns the beam out of the horizontal position until it is balanced by the moment of the original weights (*i.e.*, W at G and $2Q$ at H), tending in the opposite direction. Hence the greater the sensitiveness, the smaller must be the moment tending to restore equilibrium when w is removed, and the smaller therefore the stability of the balance.

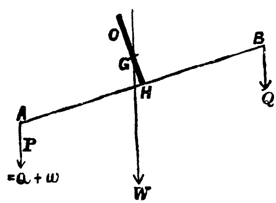

Fig. 148.

By increasing the arm a, we may obtain the same moment with a smaller weight w, and thus increase the sensitiveness, whilst, *according to the last article*, the stability would be unaltered, since it depends only on the heights GO, HO. *Practically*, however, the effect of lengthening the arms is to make the beam oscillate for a longer time

before coming to rest; hence it is impossible to attain rapid weighing with a highly sensitive chemical balance.

In a balance so constructed that G, H coincided with the fulcrum O, the beam would be in neutral equilibrium in any position with equal weights in the scale pans, but the slightest inequality in the weights would turn the beam right over. Such a balance would be highly sensitive, but would have no stability, and would therefore be practically useless.

185. **Rigidity.** — The balance must have a beam sufficiently strong not to bend under the weights which it has to carry. For this purpose a short thick beam would be preferable to a long thin one, but it would of course be less sensitive. To secure the greatest strength consistent with lightness, the beam is usually made in the form shown in Fig. 147.

186. **False balances.—Double weighing.**—A balance will evidently be false if—

(i.) The arms are of unequal length.

(ii.) The scale pans are of unequal weight.

(iii.) The beam is improperly balanced.

The method of double weighing will always give the correct weight of a body, however false the balance used. The process is as follows: — Place the body in one scale pan and balance it with suitable counterpoises (*e.g.*, small shot or fine sand) placed in the opposite pan. Now remove the body and replace it by weights sufficient to balance the counterpoises, and to bring the beam to the same position as before. These weights are evidently equal to the required weight of the body, for they act under exactly the same circumstances and produce exactly the same effect.

187. **To test the trueness of a balance,** a body is weighed first in one scale pan and then in the other. If the two observed weights are equal, the balance is true, and each is equal to the true weight of the body. If not, the balance is false, and we have the following cases to consider:—

CASE 1. **When the arms are of unequal length,** but the weights of the scale pans balance one another with the beam horizontal.

Let a, b be the lengths of the arms. Let W be the true weight of a body, P, Q the weights required to balance it when it is weighed first in one scale pan and then in the other. Then, by taking moments about the fulcrum,

we have $$W \times a = P \times b \quad \text{(i.),}$$

and $$W \times b = Q \times a \quad \text{(ii.)}$$

By multiplication, $$W^2ab = PQab, \text{ or}$$

$$W^2 = PQ.$$

Therefore $$W = \sqrt{(PQ)} \quad \text{(iii.),}$$

that is, *the true weight is the geometric mean* between the observed weights.*

The moments of the weights of the beam and scale pan do not come into the above equations, because they balance one another in the horizontal position of the beam.

To compare the lengths of the two arms, we have, by (i.), (ii.),

$$\frac{a}{b} = \frac{P}{W} \text{ and } \frac{a}{b} = \frac{W}{Q};$$

therefore $$\frac{a^2}{b^2} = \frac{WP}{WQ} = \frac{P}{Q}, \text{ or } \frac{a}{b} = \sqrt{\left(\frac{P}{Q}\right)},$$

giving the ratio a/b. When this is known, the true weight of any body may be found by weighing once and multiplying the observed weight by a/b or b/a, according to which arm the body is suspended from.

When the inequality of the arms is small (as is the case in all actual balances) it is sufficiently accurate for all practical purposes to take as the true weight the arithmetic mean $\frac{1}{2}(P+Q)$ instead of the geometric mean $\sqrt{(PQ)}$, which would be harder to calculate.

Examples.—(1) The arms of a balance are in the proportion of 9 : 10. Sugar is weighed out against $\frac{1}{2}$-lb. weight, placed first in one scale pan and then in the other. To find the total true weight of the sugar.

Since the two portions of sugar balance the $\frac{1}{2}$-lb. weight in the two pans, their actual weights are $\frac{1}{2} \times \frac{9}{10}$ and $\frac{1}{2} \times \frac{10}{9}$ lb.; therefore true weight of sugar $= \frac{1}{2}\left(\frac{9}{10} + \frac{10}{9}\right) = \frac{181}{180} = 1\frac{1}{180}$ lbs.

(2) The observed weights of a body when weighed first in one scale pan and then in the other are 101 lbs. and 99 lbs. If the discrepancy

* $\sqrt{(PQ)}$ is called the *geometric mean* between P and Q.

is due to the unequal length of the arms of the balance, find the error which would be made in taking the true weight to be 100 lbs.

The true weight $= \sqrt{(99 \times 101)} = \sqrt{9999}$ lbs.

By calculation, $\sqrt{9999} = 99 \cdot 995 = 100 - \cdot 005$.

Hence, by taking the arithmetic mean of the two weights, viz., 100 lbs., as the true weight, we should only make an error of ·005 lb., or $\frac{1}{200}$ per cent. of the whole, whereas, if we were to take either of the observed weights 101 or 99 lbs., we should make an error of 1 lb., or 1 per cent.

188. Case 2. **When the scale pans are of unequal weight,** but the arms are of equal length.

This often happens in old balances on account of the greater wear and tear of the pan with the larger surface. It is sometimes corrected by fastening a piece of lead below the lighter pan, or filing down the heavier pan. If this is not done, the beam will not be horizontal when the pans are empty, so that the error may be easily detected.

Let p, q be the weights of the two pans, P, Q the weights which, when placed in them, respectively, will balance a body whose true weight is W.

Then, since the total weights on the two sides of the beam are equal,

$$\therefore \quad P+p = W+q,$$
$$W+p = Q+q;$$
$$\therefore \quad P-W = W-Q \quad \text{or} \quad 2W = P+Q;$$
$$\therefore \quad W = \tfrac{1}{2}(P+Q);$$

that is, the true weight is the arithmetic mean of the observed weights.

189. Case 3. **When the C.G. of the beam is a little on one side,** this also will throw the beam slightly out of the horizontal, and the effect will be the same as that produced by a slight inequality in the weights of the pans, from which, however, it may be distinguished by interchanging the two pans. It is corrected by filing the beam away on the heavier side till it balances horizontally. As in Case 2, the arithmetic mean of the two weighings gives the correct weight.

It follows that *the correct weight of a body is always the arithmetic mean of its apparent weights in the two scale pans except when the inequality in the arms of the balance is considerable.* [But even in this exceptional case the method of double weighing of § 186 is correct.]

190. Roberval's Balance.—This is a letter-balance in which the scale pans are hinged to two levers *AEB*, *CFD*, which turn about their middle points *E*, *F*. When the balance is slightly displaced, one of the platforms goes up and the other goes down through exactly the same distance, and the platforms always continue to remain horizontal. Hence, if equal weights P, Q are placed anywhere on the pans, the works done by them in rising and falling will be equal and opposite, and will be the same, no matter whereabouts on the pans P, Q be placed. Therefore, by the Principle of Work, equal weights will balance one another whatever be their positions, although one may be nearer the fulcrums *E*, *F* than the other.

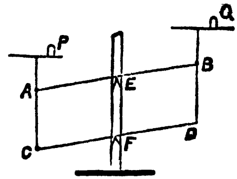

Fig. 148.

191. The common or Roman steelyard consists of a beam (*AB*, Fig. 150) moveable about a fulcrum or knife blade *C* fixed near one end *B*. From *B* is suspended the scale pan containing the body to be weighed, and a moveable weight is slid along the arm *CA* until the beam balances horizontally. The arm is graduated in such a way that the reading *P*, at which the weight rests, indicates the required weight of the body.

192. To graduate the common steelyard.—Let P denote the moveable weight. First let the scale pan be empty, and let *O* be the position of the weight P when the beam balances horizontally about *C*. Then the point *O* must be marked 0 (zero). We notice that the shorter arm *CB* and scale pan at *B* must be heavy enough for their moments about *C* to balance those of the weight of the longer arm *CA*, and of P acting at *O*.

Now let a weight W be placed in the scale pan. This weight acts on the beam at *B*, hence its moment about *C* is $W \times BC$. To balance this added moment, we must increase the moment of P by an equal and opposite

amount by moving it further away from the fulcrum. Thus, if P be the new position of P, its moment about C is increased by $P \times CP - P \times CO$, that is, by $P \times OP$.

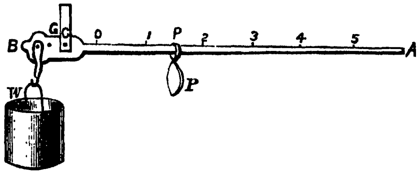

Fig. 150.

Equating the *added* moments of P, W about C, we have therefore
$$P \times OP = W \times BC,$$
or
$$OP = \frac{W}{P} \times BC.$$

Now let I denote the position of P when the unit of weight (say 1 lb.) is placed in the scale pan. Putting $W = 1$, we have
$$OI = \frac{1}{P} BC.$$

Therefore
$$OP = W \, . \, OI.$$

Hence, if $W = 2$ units, $OP = 2OI$;
if $W = 3$ units, $OP = 3OI$;
if $W = \frac{1}{2}$ unit, $OP = \frac{1}{2}OI$;

and so on. We therefore have the following rule:—

Find, **by actual trial,** *the points O, I at which P must be placed when the scale pan is empty and when it contains the unit of weight, respectively. From O measure off on OA successive multiples and submultiples of the length OI. Their extremities will be the points of graduation for the corresponding multiples and submultiples of the unit of weight.*

193. **Modifications of the common steelyard.—Weighing machines.**

In order to use the common steelyard for widely differing weights, it would be necessary either to make the arm very long, or to bring the graduations very close together: in one case the beam would be liable to bend; in the other it would lose much of its sensitiveness. For this reason most steelyards (such as the common weighbridge or

weighing machine of railway stations) carry a scale pan attached to the longer arm at A, and larger weights (*e.g.*, 28 lbs., 56 lbs., 1 cwt., &c.) are measured by a set of weights placed in this scale, smaller weights only (*e.g.*, 0 to 28 lbs.) being read off on the arm by the sliding weight. And, since the arm CA is many times longer than CB, each of the weights used in the scale at A is marked to represent a weight many times larger than itself. Thus heavy goods can be weighed by means of weights that are quite handy to lift.

This mechanical advantage is further increased in most weighing machines by means of levers placed underneath the platform carrying the goods.

In a chemical balance, small weights (milligrammes) are measured by a "rider" moved along the beam like the moveable weight in a steelyard.

*194. **The Danish steelyard** (not used in this country) has no moveable weight, but the fulcrum itself is moveable, and generally consists of a loop of string from which the beam hangs. The end A of the beam is loaded, so that when the scale is empty it balances about a point O very near that end. O is therefore the point at which the resultant weight of the beam and scale acts.

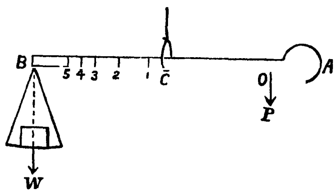

Fig. 151.

If now a series of weights of 1 lb., 2 lbs., 3 lbs., &c., be placed in succession in the scale, the fulcrum will have to be moved nearer and nearer towards the end B, for the greater the weight at B, the nearer must the balancing point be to B. By marking on the beam BA the successive points about which it balances, the steelyard will be graduated in lbs., and similarly it may be graduated in grammes or for any other set of weights.

*195. **To find a mathematical formula for the positions of the graduations.**

[In examination papers the favourite question "Show how to graduate the Danish steelyard" refers to the following mathematical formula, although, *practically*, the graduations can be found much more easily by actual trial, as explained above.]

Let W be the weight placed in the scale pan at B, and let P be the whole weight of the beam and scale pan acting at O. Then, in order

to balance, the fulcrum will have to be moved from O to a point C, such that the moments about C of P at O and W at B are equal and opposite, or

$$W \times BC = P \times CO.$$

The graduations on the Danish steelyard are not equidistant. Supposing W to be n times P, we have

$$nBC = CO;$$

$$\therefore \quad BO = BC + CO = (n+1)\,BC,$$

or

$$BC = \frac{1}{n+1}BO.$$

Putting $n = 1, 2, 3$, we see that the distances BC corresponding to weights P, $2P$, $3P$ ... are respectively $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ ... of BO. It readily follows that the distances between the graduations grow less and less towards the end B.

[The numbers 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ ... are the reciprocals of 1, 2, 3, 4 ..., and the latter numbers are in arithmetical progression. Now a series of the reciprocals of numbers in arithmetical progression is called a **harmonic progression.** Hence the distances of the successive graduations from B are in harmonic progression.]

196. **The bent lever balance** commonly used for weighing letters and small parcels consists essentially of a bent lever (usually cut out of a sheet of metal), whose arms include rather more than a right angle. The longer arm is so heavy as to rest nearly vertically in the position of equilibrium, but when a letter or parcel is suspended from the shorter arm, the moment of its weight turns the lever round. The inclination of the longer arm to the vertical in its new position of equilibrium determines the weight of the latter, and is read off by means of a pointer in front of a graduated scale arranged in various ways with which the reader is doubtless familiar. Unlike the other balances here described, the bent lever balance falls into its equilibrium position of its own accord without any adjustment of weights, &c., so that letters can be weighed very rapidly by it. A mathematical formula for the positions of the graduations can be obtained, but practically they are found by trial.

Summary of Results.

The requisites of a good balance are that it should be

(1) *true, i.e.*, the beam should be horizontal when loaded with equal weights.

Conditions.—Equal arms, scale-pans of equal weight, beam properly balanced. (§ 182.)

(2) *stable, i.e.*, the beam should return to its equilibrium position when displaced.

Conditions.—C.G. and middle point of beam *below* knife blade. (§ 183.)

(3) *sensitive*, *i.e.*, the beam sensibly deflected when weights slightly unequal.

Conditions.—Height of knife blade small, arms long. (§ 184.)

(4) *rigid*, *i.e.*, beam not bent by weights. (§ 185.)

With a *false balance* the true weight of a body may be found by *double weighing*. (§ 186.)

If the arms are unequal, W, the true weight, is the geometric mean of P, Q, the observed weights in the two pans, *i.e.*, $W = \sqrt{(PQ)}$. (§ 187.)

If the scale pans are of unequal weight, the true weight is the arithmetic mean of the observed weights, *i.e.*,

$$W = \tfrac{1}{2}(P + Q). \qquad (\S\ 188.)$$

In the *common steelyard*, a weight is moveable along the beam; graduations equidistant. (§§ 191, 192.)

In the *Danish steelyard*, the fulcrum is moveable, the weights fixed. (§ 194.)

EXAMPLES XIII.

1. Describe some form of weighing-machine, and explain carefully why the indication of the machine does not depend on the position of the body to be weighed on the platform.

2. A balance consists of a uniform rod, of length 18 ins., and weight = $\frac{1}{2}$ lb., the fulcrum being $\frac{1}{8}$ in. to one side of the C.G. of the rod. If 1 lb. be in the scale attached to the shorter arm, find how much tea a customer has weighed out to him in the other scale.

3. In a steelyard the distance of the fulcrum from the point of suspension of the weight is 1 in. and the moveable weight is 6 oz. To weigh 15 lbs. the moveable weight must be placed 8 ins. from the fulcrum. Where must it be placed to weigh 24 lbs. ?

4. The arms of a balance are 7 ins. and 8 ins. respectively. A body when suspended successively at the two extremities appears to weigh 4 lbs. and $5\frac{1}{49}$ lbs. Is the beam of the balance uniform ?

5. Can the steelyard be employed to determine whether or not the weight of a body is the same in different places ?

6. If a Danish steelyard weighs 5 lbs., and if to weigh 15 lbs. the

fulcrum must be placed 3 ins. from the point of suspension of the weight, where must it be placed in order to weigh 7 lbs.?

7. A uniform rod 2 ft. long and weighing 3 lbs. is to be used as a steelyard. The fulcrum is 2 ins. from one end of the rod, and the sliding weight is 1 lb. Find the greatest and the least weights which can be determined by this machine; and also where the sliding weight must be placed to indicate a weight of 20 lbs.

8. Write down the weights of a set of five weights capable of weighing any exact number of pounds from 1 to 31 lbs., (i.) if no weights are placed in the scale pan containing the goods, (ii.) if weights are placed in either scale pan.

9. A shopman using a common steelyard alters the moveable weight for which it has been graduated. Determine whether he cheats himself or his customers.

10. A balance has a weight attached to its beam below the centre of gravity. Is the sensibility of the balance greater when the weight is hung freely by a string, or when it is rigidly attached to the balance?

11. A Danish steelyard has the loop forming the fulcrum removed, and it is hung from a peg by two strings attached to the end of its bar. If a plumb-line be hung from the same peg, prove that the graduation on the bar which falls opposite the plumb-line will indicate the correct weight of a body placed in the scale-pan.

12. A spring-balance hangs from the shorter arm of a lever, and in weighing goods, the scale pan is raised from off the ground by pulling down the longer arm of the lever and thus lifting the balance. A person stands in the scale pan and pulls himself up in this way, and the balance indicates a weight of 8 stone. If the arms of the lever are 6 ins. and 2 ft. long respectively, find the man's true weight.

13. It is desired to change the moveable weight, 2 oz., of a steelyard for one of 1 lb. Show that there will be no necessity to alter the graduations, provided a weight equal to 7 times the weight of the steelyard is suspended at its centre of gravity.

14. There are no graduations on a certain Danish steelyard, and its weight is not known; but, by hanging up from the end A, weights P lbs. and Q lbs. in succession, it is found that the corresponding distances of the fulcrum from A are a and b ins. respectively. Find the position of the centre of gravity of the instrument, and show that its weight is

$$(bQ-aP)/(a-b) \text{ lbs.}$$

EXAMINATION PAPER VI.

1. Explain why in a common scale pan or letter balance it does not matter whereabouts on the pan the weights are placed; although they may be sometimes near, and sometimes further off, the fulcrum.

2. Describe the common steelyard, and show how to graduate it, and that the graduations are equidistant. What advantage is gained by the use of a steelyard?

3. Describe the Danish steelyard with fixed counterpoises, and show that the distances between the points of graduation on the load arm form a harmonical progression.

4. Explain the method of double weighing in a balance, and show that any inequality in the arms of the balance will not affect the accuracy of the result obtained.

5. A steelyard is correctly graduated when new; but, by the wearing away of the rod, the weight of the rod and the position of its centre of gravity are slightly changed. It is found that a body appearing to weigh 2 lbs. in reality weighs 2 lbs. and $\frac{1}{4}$ oz. Find the true weight of a body appearing to weigh 10 lbs.

6. Find an expression for the whole amount of work done in raising several weights through different heights.

7. A uniform beam weighs 1000 lbs. and is 20 ft. long. It hangs by one end, round which it can turn freely. How many foot-pounds of work must be done to raise it from its lowest to its highest position?

8. A thread 9 ft. long has its ends fastened to the ends of a weightless rod 6 ft. long. The rod is supported in such a manner as to be capable of turning freely round a point 2 ft. from one end. A weight is placed on the thread, like a bead on a string. Give a diagram showing the position in which the rod will come to rest.

9. Find the centre of gravity of equal masses placed at each of five of the corners of a regular hexagon.

10. Two equal heavy spheres of 1 in. radius are in equilibrium within a smooth spherical cup of 3 ins. radius. Show that the thrust between the cup and one of the spheres is double the thrust between the two spheres.

CHAPTER

ON TRIGONOMETRY AND MENSURATION.

1. **Trigonometry** is that branch of mathematics which deals with **angles.**

2. The **units of angular measure** are the subdivisions of a right angle defined as follows:—

1 right angle = 90 **degrees,** denoted by **90°**;
1 degree or 1° = 60 **minutes,** denoted by **60′**;
1 minute or 1′ = 60 **seconds,** denoted by **60″**.

3. **Particular right-angled triangles.**

By Euc. I. 47, $AB^2 + BC^2 = AC^2$, where $\angle ABC = 90°$.

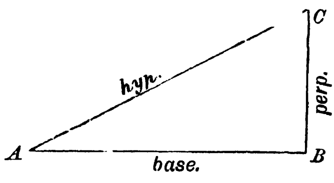

Fig. 1.

Thus *the numbers* **3, 4, 5,** or their multiples (*e.g.*, 6, 8, 10), *are proportional to the sides of a right-angled triangle, for* $9 + 16 = 25$, *i.e.* $3^2 + 4^2 = 5^2$. These numbers should be remembered. Another such set is 5, 12, 13, for $5^2 = 25 \times 1 = (13 + 12)(13 - 12) = 13^2 - 12^2$, and $\therefore\ 5^2 + 12^2 = 13^2$.

4. The **trigonometrical ratios** of an angle.

If AX be a fixed straight line, and we imagine a straight line AC, initially coincident with and equal to AX, to revolve round A in the direction indicated by the arrow, *i.e.* counter-clockwise, it coincides with AY when it has turned through 90° and is perpendicular to AX,

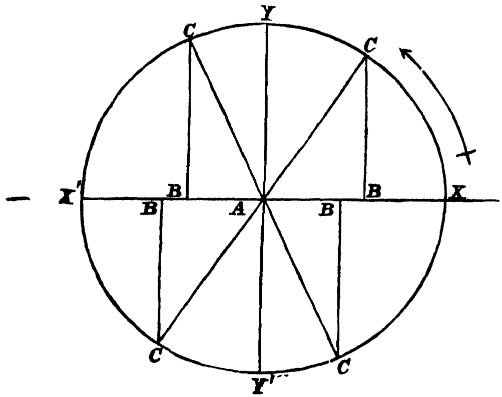

Fig. 2.

with AX' in AX produced when it has turned through 180°, with AY' in AY produced when it has turned through 270°, and again with AX when it has turned through 360°. Thus the point C traces out the circumference of a circle, and the line AC traces out angles from 0° onwards.

Draw CB perpendicular to AX or AX produced.

Then in *every* position of the point C,

BC, the side of $\triangle ABC$ opposite to the $\angle$ at A, is termed the *perpendicular* of $\angle BAC$;

AB, the side of $\triangle ABC$ adjacent to the $\angle$ at A, is termed the *base*;

and AC, the hypotenuse of $\triangle ABC$, is termed the *hypotenuse*.

If A be the measure of $\angle BAC$, the trigonometrical ratios are defined as follows:—

the ratio	$\frac{BC}{AC}$ or	$\frac{\text{perpendicular}}{\text{hypotenuse}}$	is called the	**sine**	of the angle A and is written	**sin** A;
,,	,, $\frac{AB}{AC}$,,	$\frac{\text{base}}{\text{hypotenuse}}$	,,	**cosine**	,, ,,	**cos** A;
,,	,, $\frac{BC}{AB}$,,	$\frac{\text{perpendicular}}{\text{base}}$	,,	**tangent**	,, ,,	**tan** A.

5. The order of the letters is important, especially when the lines represent velocities or forces. Thus $\sin BAC$ must be written $= BC \div AC$ and not $= CB \div AC$.

6. If C be between	X and Y,	$\angle XAC$ is said to be in the	1st	quadrant;
,, ,,	Y ,, X'	,, ,, ,,	2nd	,,
,, ,,	X' ,, Y'	,, ,, ,,	3rd	,,
,, ,,	Y' ,, X	,, ,, ,,	4th	,,

7. **Positive and negative trigonometrical ratios.**

The line AB when measured in the direction AX is considered positive; when in direction AX' it is considered negative. In the same manner the line BC when measured in direction AY is positive, and when in direction AY' negative.

The hypotenuse AC is considered positive in all positions.

Thus AB is positive in the first and fourth quadrants, and is negative in the second and third quadrants; BC is positive in the first and second quadrants and negative in the third and fourth quadrants.

Therefore angles between 0° and 90° have their
sine positive (+), **cosine positive** (+), **tangent positive** (+),
and **angles between 90° and 180°** have their
sine positive (+), **cosine negative** (−), **tangent negative** (−).

8. **To find the trigonometrical ratios of an angle of 45°.**

Draw a square $ABCD$, and draw the diagonal AC. (The student can supply the figure.)

Then $\angle BAC$ = half a right angle $= 45°$,

$\angle CBA$ = a right angle $= 90°$;

$\therefore \quad AC^2 = AB^2 + BC^2.$ (Euc. I. 47.)

Also $AB = BC$;

$$\therefore \quad AC^2 = 2AB^2 = 2BC^2;$$
$$\therefore \quad AC = \sqrt{2} . AB = \sqrt{2} . BC;$$
$$\therefore \ \sin 45^\circ = \frac{BC}{AC} = \frac{1}{\sqrt{2}}, \ \cos 45^\circ = \frac{AB}{AC} = \frac{1}{\sqrt{2}}, \ \tan 45^\circ = \frac{BC}{AB} = 1.$$

COROLLARY.—If the angles of a triangle be 45°, 45°, and 90°, the sides are proportional to 1, 1, and $\sqrt{2}$.

9. **To find the trigonometrical ratios for an angle of 30°.**

Draw an equilateral triangle ABC. Join A to D, the middle point of BC. Then the triangles ABD, ACD are equal in every respect.

But the three angles of an equilateral triangle are all equal, and are together = two right angles = 180°; therefore each = 60°.

$$\therefore \quad DBA = 60^\circ;$$
$$\therefore \quad \angle DAB = 30^\circ.$$

Also $\angle ADB = \angle ADC = 90^\circ$;

$$\therefore \quad AB^2 = AD^2 + DB^2. \quad \text{(Euc. I. 47).}$$

But $AB = CB = 2DB$;

$$\therefore \quad 4DB^2 = AD^2 + DB^2 \quad \text{or} \quad AD^2 = 3DB^2;$$
$$\therefore \quad AD = \sqrt{3} . DB;$$

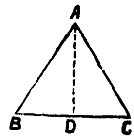

Fig. 3.

$$\therefore \ \sin 30^\circ = \frac{DB}{AB} = \frac{1}{2}, \ \cos 30^\circ = \frac{AD}{AB} = \frac{\sqrt{3}}{2}, \ \tan 30^\circ = \frac{DB}{AD} = \frac{1}{\sqrt{3}}.$$

COROLLARY.—If the angles of a triangle be 30°, 60°, and 90°, the sides opposite these angles are proportional to 1, $\sqrt{3}$, and 2.

10. **To find the trigonometrical ratios for an angle of 60°.**

Take the figure of § 9. Then $\angle CBA = 60^\circ$. Therefore

$$\sin 60^\circ = \frac{AD}{AB} = \frac{\sqrt{3}}{2}, \ \cos 60^\circ = \frac{BD}{AB} = \frac{1}{2}, \ \tan 60^\circ = \frac{AD}{BD} = \sqrt{3}.$$

11. **To find the trigonometrical ratios of an angle of 0°.**

If the angle BAC is *zero*, AC will coincide with AB, and C with B, and the perpendicular BC will vanish.

$$\therefore \quad AB = AC \quad \text{and} \quad BC = 0;$$

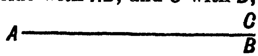

Fig. 4.

$$\therefore \ \sin 0^\circ = \frac{BC}{AC} = 0, \ \cos 0^\circ = \frac{AB}{AC} = 1, \ \tan 0^\circ = \frac{BC}{AB} = 0.$$

12. To find the trigonometrical ratios of an angle of 90°.

Let DAC = a right angle = 90°. Then, if CB is drawn perpendicular on AD, CB will coincide with CA, and B with A;

$\therefore\ BC = AC$ and $AB = 0$;

$$\therefore\ \sin 90^\circ = \frac{BC}{AC} = 1,\quad \cos 90^\circ = \frac{AB}{AC} = 0,$$

$$\tan 90^\circ = \frac{BC}{AB} = \frac{BC}{0} = \infty,$$

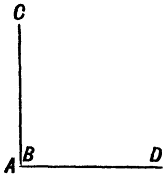

Fig. 5.

where ∞ stands for "infinity." For, if $BC = a$, we shall find that 0 will divide into a any number of times and still leave a remainder a; hence $a \div 0$ must be greater than any number however great, and we write this fact thus: $a \div 0 = \infty$.

[It is also useful to observe that $a \div \infty = 0$ unless $a = \infty$.

13. Table.—The trigonometrical ratios for the above angles are given in the following table, *which should be remembered.*

Angle	0°	30°	45°	60°	90°
sin =	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos =	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan =	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

If there be any difficulty in fixing these in the memory, it may be noticed that, for the common angles, 0°, 30°, 45°, 60°, 90°, the sines are the *square roots* of

$$\frac{0}{4},\ \frac{1}{4},\ \frac{2}{4},\ \frac{3}{4},\ \frac{4}{4};$$

the cosines are the *square roots* of

$$\frac{4}{4},\ \frac{3}{4},\ \frac{2}{4},\ \frac{1}{4},\ \frac{0}{4};$$

and the tangents are the square roots of the first series of numerators divided by the corresponding numerators of the second series.

Note that the series of values of the cosines is the same as the series for sines written backwards; and each tangent is the corresponding sine $\div$ cosine.

14. **To prove that** $\dfrac{\sin A}{\cos A} = \tan A$.

$$\sin A \div \cos A = \frac{BC}{AC} \div \frac{AB}{AC} = \frac{BC}{AC} \times \frac{AC}{AB} = \frac{BC}{AB} = \tan A.$$

15. **To prove that** $(\sin A)^2 + (\cos A)^2 = 1$.

By Euc. I. 47, $AC^2 = AB^2 + BC^2$;

$$\therefore \quad 1 = \frac{AC^2}{AC^2} = \frac{AB^2}{AC^2} + \frac{BC^2}{AC^2} = \left(\frac{AB}{AC}\right)^2 + \left(\frac{BC}{AC}\right)^2 = (\sin A)^2 + (\cos A)^2.$$

This is written $\sin^2 A + \cos^2 A = 1$.

16. **The trigonometrical ratios depend only on the angle** and not on the size of the right-angled triangle constructed in defining them. Thus $\sin 30° = \frac{1}{2}$, and this tells us that in *any* right-angled triangle having an angle of 30°, the perpendicular is $\frac{1}{2}$ the hypotenuse. The same thing is illustrated by the other trigonometrical ratios tabulated in § 13, and it can be proved generally for *any* angle by means of Euc. VI. 4.

17. The angle $180° - A$ is called the **supplement** of the angle A.

Let $\angle BAC = A$. Produce BA to B', and make $\angle B'AC' = \angle BAC$.

Then $\angle BAC' = 180° - A$

$= \text{supplement of } A$.

Take $AC' = AC$,

and drop the perpendiculars CB, $C'B'$.

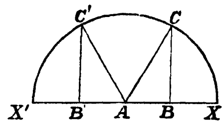

Fig. 6.

Then the triangles BAC, $B'AC'$ are equal in all respects, and AB, AB' are in opposite directions;

$$\therefore \quad B'C' = BC, \quad \text{and} \quad AB' = -AB;$$

$$\therefore \quad \sin(180° - A) = \sin BAC' = \frac{B'C'}{AC'} = \frac{BC}{AC} = \sin A,$$

$$\cos(180° - A) = \cos BAC' = \frac{AB'}{AC'} = -\frac{AB}{AC} = -\cos A,$$

$$\tan(180° - A) = \tan BAC' = \frac{B'C'}{AB'} = \frac{BC}{-AB} = -\tan A.$$

Hence $\sin 120° = \sin(180° - 60°) = \sin 60° = \frac{1}{2}\sqrt{3}$,

and $\cos 120° = \cos(180° - 60°) = -\cos 60° = -\frac{1}{2}$.

Similarly the sine, cosine, and tangent of $135° (= 180° - 45°)$, $150° (= 180° - 30°)$, and $180° (= 180° - 0°)$ can be found.

18. Combining these results with those of § 13, we may extend our table as follows:—

Angles

$0°, \quad 30°, \quad 45°, \quad 60°, \quad 90°, \quad 120°, \quad 135°, \quad 150°, \quad 180°$

sine

$$\sqrt{\frac{0}{4}}, \sqrt{\frac{1}{4}}, \sqrt{\frac{2}{4}}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{4}}, \sqrt{\frac{3}{4}}, \sqrt{\frac{2}{4}}, \sqrt{\frac{1}{4}}, \sqrt{\frac{0}{4}};$$

cosine

$$\sqrt{\frac{4}{4}}, \sqrt{\frac{3}{4}}, \sqrt{\frac{2}{4}}, \sqrt{\frac{1}{4}}, \sqrt{\frac{0}{4}}, -\sqrt{\frac{1}{4}}, -\sqrt{\frac{2}{4}}, -\sqrt{\frac{3}{4}}, -\sqrt{\frac{4}{4}};$$

It is, however, better not to remember the ratios of angles between 90° and 180°, but to obtain them when required, from the ratios of their supplements, by the formulæ of § 17.

19. If the angles of a triangle be 30°, 30°, and 120°, the sides are proportional to 1, 1, and $\sqrt{3}$.

Let ABC be the triangle, D the middle point of the longest side BC.

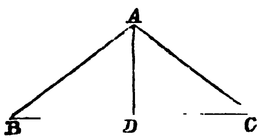

Fig. 7.

Then AD is perpendicular to BC. Hence

$$\angle ADB = 90°, \quad \text{and} \quad \angle ABD = 30°$$

$$\therefore \quad BD = \tfrac{1}{2}\sqrt{3} \,.\, BA \quad (\S\ 9, \text{Cor.});$$

$$\therefore \quad BC = 2BD = \sqrt{3} \,.\, BA.$$

20. Results in Mensuration.

The following facts in Solid Geometry and Mensuration are assumed. The references given below are to the articles in Briggs and Edmondson's *Mensuration of the Simpler Figures*, where the reader will find the properties in question fully proved. Proofs of them are also given in most elementary treatises on Solid Geometry. The *results* alone need be remembered:—

(1) **The area of a triangle**

$$= \frac{1}{2}\,(base) \times (altitude). \qquad (\S\ 45.)$$

(2) **The area of a trapezoid** (*i.e.* a quadrilateral with two sides parallel) $= (its\ height) \times (\tfrac{1}{2}\ sum\ of\ parallel\ sides)$. (§ 49.)

(3) **The length of the circumference of a circle** of radius r

$$= \pi \times (diameter)$$

$$= 2\pi r; \qquad (\S\ 57.)$$

where the Greek letter π ("**pi**") stands for a certain "incommensurable" number (that is, a number which cannot be expressed as an exact arithmetical fraction), whose value lies between 3·141592 and 3·141593. The following approximate values should be remembered and used, unless otherwise stated.

$$\pi = \frac{22}{7}, \quad \text{for all } rough \text{ calculations};$$

$$\pi = 3{\cdot}1416, \text{ more approximately.}$$

(4) **The area of the circle**

$$= \frac{1}{2}\,(radius) \times (circumference)$$

$$= \pi r^2. \qquad (\S\ 58.)$$

(5) **The volume of a pyramid**

$$= \frac{1}{3}\,(height) \times (area\ of\ base)$$

$$= \frac{1}{3}\,hA. \qquad (\S\ 105.)$$

the height h being the perpendicular from the vertex on the plane of the base, and A the area of the base.

(6) **The area of the curved surface of a cylinder,** whose height is h and the radius of whose base is r,

$$= (\textit{height}) \times (\textit{circumference of base})$$
$$= 2\pi rh. \qquad (\S\ 115.)$$

(7) **The volume of the cylinder**

$$= (\textit{height}) \times (\textit{area of base})$$
$$= \pi r^2 h. \qquad (\S\ 116.)$$

(8) **The area of the curved surface of a right circular cone,** whose height is h and the radius of whose base is r,

$$= \frac{1}{2}(\textit{circumference of base}) \times (\textit{length of slant side})$$
$$= \pi r \sqrt{(h^2+r^2)}\,; \qquad (\S\ 117.)$$

a *slant side* being a line drawn from the vertex to a point in the circumference of the base.

(9) **The volume of the cone**

$$= \frac{1}{3}(\textit{vol. of cylinder of same base and height})$$
$$= \frac{1}{3}\pi r^2 h. \qquad (\S\ 118.)$$

(10) **The area of the surface of a sphere** of radius r

$$= 4 \textit{ times area of circle of same radius}$$
$$= 4\pi r^2. \qquad (\S\ 126.)$$

(11) **The volume of the sphere**

$$= \frac{1}{3}(\textit{radius}) \times (\textit{surface})$$
$$= \frac{4}{3}\pi r^3. \qquad (\S\S\ 127,\ 128.)$$

ANSWERS.

EXAMPLES I. (PAGES 19, 20.)

1. About $1\frac{4}{5}$ lbs. wt. **2.** 16 lbs. **3.** 10 lbs. and 15 lbs.

4. (i.) $8AP$, where P divides BC, so that $3BP = 5CP$.
(ii.) $5PA$, where P divides BC, so that $2BP = 3CP$.
(iii.) $2AP$, where BC is produced to P, and $2CP = BC$.
(iv.) PA, where BC is produced to P, and $CP = 4BC$.
(v.) $9AP$, where BC is produced to P, and $2BC = 9CP$.
(vi.) PA, where BC is produced to P, and $CP = 8BC$.

5. 13 lbs. wt. **6.** $\sqrt{3}P$, at right angles to $2P$ between $2P$ and $3P$.

7. See § 19; $2\sqrt{2}$ in a north-easterly direction. **8.** 90°.

10. Zero. **11.** In P, where $(m-k)BP = (k-l)CP$.

12. Draw AF parallel to ED to meet BD in F.

EXAMPLES II. (PAGES 35, 36.)

1. (i.) $2\sqrt{3}$ lbs.; 2 lbs. (ii.) 8 oz.; 8 oz.
(iii.) 5 kilog.; $5\sqrt{3}$kilog. (iv.) 0; 3 tons.
(v.) -6 grs.; $6\sqrt{3}$ grs. (vi.) $-\frac{5}{2}\sqrt{2}$ lbs.; $\frac{5}{2}\sqrt{2}$ lbs.
(vii.) $-4\sqrt{3}$ cwt.; 4 cwt. (viii.) -4 mgm.; 0.
(ix.) 6 stone; 0.

2. $10\sqrt{5}$ lbs.; inclined to vertical at an angle whose cosine is $2/\sqrt{5}$.

3. (i.) 1 lb. along each. (ii.) $\frac{3}{2}$ lbs. along each.

4. (i.) 7 lbs. (ii.) 26 grammes.
(iii.) 1 ton. (iv.) 5 kilog.
(v.) 5 cwt. (vi.) $2\sqrt{7}$ lbs.
(vii.) $5\sqrt{3}$ lbs. (viii.) $4\sqrt{10+3\sqrt{3}}$ mgm.
(ix.) $2\sqrt{13-6\sqrt{3}}$ oz.

5. $2\sqrt{39}$ lbs.

6. 12 lbs. opposite to the 10-lb. force, and $2\sqrt{3}$ lbs. at right angles to this on the side remote from the 4-lb. force.

8. $2\sqrt{35-18\sqrt{2}}$ lbs. between north and west.

9. $7\sqrt{3}$ lbs.

10. $\sqrt{3}$ lbs.

12. If P is $> Q$, Q and $\sqrt{RR'}$ are perpendicular to each other.

13. $-\frac{5}{9}$. **14.** $-\frac{1}{7}$. **15.** $5(\sqrt{3}+1)$ lbs. ; $5(\sqrt{3}-1)$ lbs.

16. 16·93 lbs., 10·65 lbs. ; 4·62 lbs., ·78 lb.

Examination Paper I. (Page 37.)

1. See § 5. **2.** See §§ 12, 13. **3.** See § 16.

4. $5\dfrac{AP}{AB}$, in direction parallel to AP, where BC is produced to P, and $2BC = 5CP$.

5. See §§ 18, 20, 21. **6.** See § 24. **7.** See §§ 27, 29.

8. Draw DF parallel to EA to meet AB in F.

9. See § 31. **10.** See § 34.

Examples III. (Pages 49, 50.)

1. (i.) $7\frac{1}{2}$ lbs. ; 6 lbs. (ii.) $32\frac{1}{2}$ lbs. ; 30 lbs. (iii.) $8\frac{3}{7}$ tons ; $8\frac{2}{5}$ tons. (iv.) $45\frac{1}{3}$ kilog. ; 40 kilog.

2. (i.) 60 ft.-lbs. (ii.) 390 ft.-lbs. (iii.) 630 ft.-tons. (iv.) 1360 kilogram-metres.

3. (i.) $\frac{5}{3}\sqrt{3}$ tons ; $2\frac{1}{2}$ tons. (ii.) 28 lbs. ; $14\sqrt{2}$ lbs. (iii.) $10\sqrt{3}$ kilog. ; $5\sqrt{3}$ kilog. The reactions are respectively (i.) $\frac{10}{3}\sqrt{3}$ tons ; $\frac{5}{2}\sqrt{3}$ tons. (ii.) $28\sqrt{2}$ lbs. ; $14\sqrt{2}$ lbs. (iii.) 20 kilog. ; 5 kilog.

4. (i.) 4200 ft.-lbs. (ii.) $42\sqrt{2}$ ft.-lbs. (iii.) $\frac{15}{2}\sqrt{3}$ kilogram-metres.

5. (i.) $6\frac{3}{4}$ lbs. (ii.) $5\frac{2}{5}$ lbs. ; $13\frac{1}{2}$ ft.-lbs. **6.** $6\sqrt{2}$ lbs.

7. $\frac{10}{3}\sqrt{3}$ lbs. **8.** $\frac{1}{2}$ ton. **10.** 5 lbs. ; $\frac{5}{3}\sqrt{3}$ lbs.

11. Along the plane, $W\sin\alpha$; the force must act towards the plane at an angle of $(90°-\alpha)$ with it.

12. $T = \dfrac{W}{2} = \dfrac{W\sqrt{3}}{2}$. **13.** $\left(\dfrac{AB}{AC}\right)^2 P$.

14. $28\sqrt{2}$ lbs. ; when the angle between the two parts of the string is equal to or greater than 120°.

15. Anywhere in the tube.

17. The tension in each string equals any one of the weights.

Examples IV. (Pages 61, 62.)

1. See § 58. **2.** $\frac{3}{8}$ lb.

5. The rod is inclined at 30° to the vertical; 32 lbs.

6. $1\frac{1}{2}$ ins.; 60° with horizontal.

7. $\frac{\sqrt{7}}{4}W$, $\frac{\sqrt{3}}{4}W$, where W is the weight of the rod.

8. $10\sqrt{2}$ lbs.; 10 lbs. **9.** $\frac{\sqrt{3}}{2}W$; $\frac{1}{2}W$. **11.** $\frac{\sqrt{3}}{2}$ lb.; $\frac{1}{2}$ lb.

12. Tension in each string $= \frac{W}{2}(\sqrt{3}-1)$.

Examination Paper II. (Page 63.)

1. See § 52. **2.** See § 53. **3.** See § 57.

4. See § 44. **5.** 15 lbs. **6.** See § 47.

7. $2\sqrt{3}$ lbs., at right angles to the 5-lb. force between the 5-lb. and 7-lb. forces.

8. See § 53. **9.** $Q = bP/a$, $R = cP/a$.

Examples V. (Pages 74, 75.)

1. (i.) $2\frac{1}{2}$, (ii.) $\frac{5}{4}\sqrt{2}$, (iii.) $\frac{5}{4}\sqrt{2}$, (iv.) $\frac{5}{4}\sqrt{3}$, in ft.-lb. units.

2. $\frac{3}{4}\sqrt{3}AB$; $2\sqrt{3}AB$; $\frac{5}{2}AB$.

3. Towards the side on which A lies. **4.** See §§ 60, 61; $\sqrt{3} : 4$.

6. About one end, 0, +4, +18, −32; about other end, +8, −12, −6, 0; about middle point, +4, −4, +6, −16. The sum of the moments is −10, round any one of the points.

7. 0, $\frac{3\sqrt{3}}{2}AB$, $5\sqrt{3}AB$, $\frac{9\sqrt{3}}{2}AB$, 0.

8. $8\sqrt{3}$, $10\sqrt{3}$; at P in BC produced, such that $CP = 16$ ft

9. $(P-Q)a / \sqrt{P^2+Q^2}$, where a is the length of the side.

10. Two straight lines parallel to the resultant at a distance of $\frac{1}{4}$ on each side of it.

11. The resultant passes through A.

12. The moments and sum of the moments round the four corners are respectively

(i.) −3, −4, +10, +12; +15. (ii.) +6, −4, −5, +12; +9.

(iii.) +6, +8, −5, −6; +3. (iv.) −3, +8, +10, −6; +9.

13. Any point in a line parallel to CD and distant from it $2a$ on the side opposite to AB, where a is the length of a side of the square.

Examples VI. (Pages 89, 90.)

1. (i.) 4 lbs.; $1\frac{1}{2}$ ft.*, $\frac{1}{2}$ ft. (ii.) 12 lbs.; 21 ins., 15 ins.
(iii.) 12 lbs.; $31\frac{1}{2}$ ins., $4\frac{1}{2}$ ins. (iv.) 14 lbs.; 30 ins., 12 ins.
(v.) $\frac{3}{4}$ ton; 4 ins., 2 ins. (vi.) 1 kilog.; 6 cm., 4 cm.

2. (i.) 2 lbs.; 3 ft., 1 ft. (ii.) 2 lbs.; $10\frac{1}{2}$ ft., $7\frac{1}{2}$ ft.
(iii.) 9 lbs.; $3\frac{1}{2}$ ft., $\frac{1}{2}$ ft. (iv.) 6 lbs.; 70 ins., 28 ins.
(v.) $\frac{1}{4}$ ton; 12 ins., 6 ins. (vi.) 200 gms.; 30 cm., 20 cm.

3. $5\frac{15}{19}$ ft. from the 8-lb. end; 19 lbs.

4. 1 unit, at a distance of 1 ft. from the force of 2 units, and 2 ft. from the force of 1 unit.

5. The bar balances about a point distant $1\frac{1}{4}$ ft. from the boy.

6. See § 78. **7.** $68\frac{2}{11}$ lbs., $81\frac{9}{11}$ lbs. **8.** 25 lbs.

9. $P = 8$ lbs., $Q = 9$ lbs. **10.** 12 lbs.

13. 6 lbs.; 8 ft. from the table. **14.** $(P^2 - Q^2)/P$, if P be the greater.

Examples VII. (Pages 108–110.)

1. 15 lbs. **2.** $\frac{2}{3}$ cwt., acting downwards. **3.** 8 lbs., 10 lbs.

5. 3 ft. from the man bearing 94 lbs. of the whole weight.

6. 10 lbs., $1047\frac{13}{21}$ ft.-lbs. **8.** $94\frac{1}{2}$ lbs. **9.** $4\frac{1}{3}$ lbs.

10. $P = 9$ lbs., $Q = 15$ lbs. **12.** $\frac{1}{3}AB$ from A; pressure $= \sqrt{3}P$.

13. Tension of thread $= W$; pressure at $C = \sqrt{3}W$ perpendicular to AB.

14. 3 lbs.; 3 ft.

15. Radius of wheel is four times radius of axle; weight of wheel and axle and weight of man. **16.** See § 98.

Examination Paper III. (Page 111.)

1. See § 61. **2.** See § 69. **3.** See § 78.

4. (*a*) 300 ft.-lb. units.; 750 ft.-lb. units, in opposite sense.
(*b*) 450 ft.-lb. units; to the side of AC on which B lies.

5. See § 80. **6.** 21 lbs. on A, 9 lbs. on B; 10 lbs. more in each case.

7. See §§ 89–92. **9.** See § 98. **10.** 9 lbs.

Examples VIII. (Pages 126, 127.)

1. $\frac{1}{16}$, 20 lbs. **2.** $\frac{1}{31}$, 14 lbs. **3.** 6 cwt. **4.** Three moveable pulleys.

5. 22 lbs. **6.** Light in the first system, heavy in the third.

7. 7 lbs. **8.** 8 lbs., 4 lbs., 15 lbs. **9.** 16 lbs., 4 lbs., 7 lbs.

---

* In Examples 1, 2, the distance of the resultant from the smaller component is given first.

10. 250 lbs., ; 250 lbs., 500 lbs., 1000 lbs., 2000 lbs.

11. $7\frac{1}{3}$ stone, $18\frac{2}{3}$ stone. **14.** 9 stone.

Examples IX. (Pages 136, 137.)

2. 15 lbs. parallel to *AC*, and at a distance from it equal to $\frac{2}{3}AB$.

3. See § 125. **4.** See § 124.

5. (*a*) $\frac{2P}{BC}$. area of triangle *ABC*.

(*b*) $2P$ acting along the line bisecting *AC* and *BC*.

6. Complete parallelogram *ABDC*. Required force equals $\frac{1}{4}AD$, and acts parallel to it through *B* or *C*.

7. 1580 lbs. nearly. **8.** $\frac{33}{49}$ in. **9.** $\frac{480}{297}$ lbs.

10. $2\pi nM$; $942\frac{6}{7}$ ft.-lbs. **11.** See § 126. **14.** 224.

Examination Paper IV. (Page 138.)

1. See §§ 98, 114. **2.** See §§ 117, 124. **3.** See §§ 106, 108.

4. 622 lbs. **5.** See § 129. **6.** $3\frac{57}{245}$ ins.; $\frac{18}{35}$ in.

8. 4 lbs. wt., acting parallel to, and in the same direction as, the given force of 4 lbs., and at a distance of $1\frac{1}{2}$ ft. from it.

9. 3 lbs., mechanical advantage = 6.

10. The radii should be as the numbers 1, 2, 3 ..., or as 1, 3, 5 ..., according as the string is fastened to the lower or to the upper block.

Examples X. (Pages 148–150.)

1. 3 ft. from the man carrying 71 lbs.

2. 21 lbs., acting between the 7-lb. and 9-lb. forces at a distance of 3 ft. from the latter.

3. When the 100 lbs. is placed on the end furthest from a prop, the pressures are 225 lbs. downwards on the nearer prop, and 5 lbs. upwards on the further prop; when the 100 lbs. is placed on the other end, the downward pressures are 170 lbs. on the nearer and 50 lbs. on the further prop.

4. $17\frac{1}{2}$ lbs.

5. 11 lbs., acting in the same direction as the 7-lb. force, at a distance of $\frac{3}{11}$ ft. from the end where the 3 lbs. is.

6. $3\frac{1}{2}$ cwt. on each. **7.** 3 ft. from the first prop. **8.** 228.

9. (*a*) $36\frac{2}{5}$ lbs., $54\frac{3}{5}$ lbs. (*b*) $78\frac{2}{5}$ lbs., $12\frac{3}{5}$ lbs. (*c*) $-5\frac{3}{5}$ lbs., $96\frac{3}{5}$ lbs.

10. (i.) The centre of the triangle. (ii.) The vertex of the equilateral triangle described on the other side of the base opposite the point of application of the unlike force.

11. Take F in AB, so that $2AF = FB$; the centre of the forces is at G in FC, where $9FG = 2GC$.

12. Produce BA to F, so that $AF = AB$; the centre of the forces is at G in FC, where $3FG = 2GC$.

13. 540. **14.** 4 ins. **15.** $\frac{2}{5}$ in. nearer the centre.

16. It is moved from the centre of the square to the corner opposite the point of application of the reversed force.

17. At an infinite distance.

18. $\frac{7AB}{3}$, $\frac{AB}{3}$, from AB and AD respectively.

19. At K in EG, such that $OK = 8$ ins.

20. Between the strings over the fixed pulley and the next, and $\frac{11}{15}$ ft. from the former.

Examples XI. (Pages 170, 171.)

1. 6 lbs. **2.** See § 150. **3.** 12 lbs.; at middle point of rod.

4. 7 oz.; $9\frac{1}{7}$ in. from A. **5.** See § 149.

6. 2 oz.; 12 ins. **7.** See § 150.

8. See § 156; if the 8-in. side was originally vertical, the block will topple when the plank forms an inclined plane, such that height $= \frac{5}{8}$ base.

9. The straight line bisecting AB and CD.

10. The sphere containing the lead has *two* positions of equilibrium only when placed on a table—one stable, the other unstable; the other sphere is stable in all positions.

12. 450 sq. ft.

Examples XII. (Pages 187–189.)

1. Suspend it by *three* strings from one point from which is hung a plumb-line. This cuts the plane of the hoop in the required C.G.

2. 2 ft. from vertex; 8 ins. from vertex.

3. $\frac{13}{22}AB$, $\frac{8}{11}AB$, from AB and AD respectively.

4. 4 ins. from AB, $3\frac{1}{2}$ ins. from AD. **5.** $\frac{1}{12}AC$ from O towards C.

7. $\frac{23}{42}AC$ from A towards C. **8.** See § 177.

11. $\frac{1}{9}AB$ from O towards middle point of CD. **12.** See § 169.

13. 2·55 ins. from base.

16. $\frac{Wl}{2}$, where W is the weight and l the length of the chain.

17. One-third of radius from centre of board.

20. The masses are proportional to BC, CA, and AB. **22.** $\frac{30+\sqrt{6}}{48}a$.

23. One-third of the diagonal from the point where the weight is suspended.

24. At the middle point of the common side.

25. 16 ft.-lbs.; $5\frac{1}{3}$ ins. **26.** 192,500,000 ft.-lbs. **28.** $\frac{2}{3}a$.

Examination Paper V. (Page 190.)

1. See § 135. **2.** See § 150. **3.** See § 173. **4.** See § 169.

6. See § 160. **7.** $7\frac{5}{19}$ ins. from A; 46 ins. from A in BA produced.

8. 2 ins. from centre of box towards the middle point of the vertical face to which the hinges are attached.

9. See § 176. **10.** Same as C.G. of original triangle.

Examples XIII. (Pages 202, 203.)

1. See § 181. **2.** $\frac{141}{146}$ lbs. **3.** 32 ins. from fulcrum.

4. Yes. **5.** No. **6.** 5 ins. from the weight.

7. 15 lbs., 26 lbs.; at middle of rod.

8. 1 lb., 2 lbs., 4 lbs., 8 lbs., 16 lbs.; 1 lb., 3 lbs., 9 lbs., 27 lbs.

9. With diminished weight he cheats his customers, with increased weight he cheats himself.

10. When the weight is hung by a string. **12.** 10 stone.

Examination Paper VI. (Page 204.)

1. See § 182.

2. See §§ 191, 192; only one weight used, and this a comparatively small one.

3. See §§ 193, 194. **4.** See § 186. **5.** 10 lbs. $2\frac{1}{2}$ oz.

6. See § 175. **7.** 20,000 ft.-lbs.

8. If ACB be the thread and C the position of the weight, $AC = 6$ ft., $CB = 3$ ft.

9. $\frac{1}{5}a$ from centre of hexagon to corner opposite to that at which no weight is situated, a being side of hexagon.

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